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Abel fractional differential equations using Variation of parameters method

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Abstract

The Variation of Parameters Method (VPM) is utilized throughout the research to identify a numerical model for a nonlinear fractional Abel differential equation (FADE). The approach given here is used to solve the initial problem of fractional Abel differential equations. There is no conversion, quantization, disturbance, structural change, or precautionary concerns in the proposed method, although it is easy with numerical solutions. The measured values are graphed and tabulated to be compared with the numerical model.
Keywords: Abel fractional differential equations; Reimann-liouville fractional integral; Reimann-liouville fractional derivative; variation of parameters method.[1]

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1 Introduction

Abel differential equations is well established, perturbed of implementations in structure and function in dynamics, linear systems with Stochastic, modeling technique, and linear algebra. Numerous research has been done mostly on methods of the Abel differential equations. we suggest several methods, including the iterative method of the Abel differential equation achieved from the variation iteration method. We consider the following nonlinear FADE:

$$D^{\beta}f(x) = P_1 f^3(x) + P_2 f^2(x) + P_3 f(x) + P_4,\ \ 0 < \beta \leq 1,\ \ 0 \leq x \leq R. \quad (1)$$

with the initial condition

$$f^k(0) = u_k, k = 1, 2, 3, ...n - 1. \quad (2)$$

where P_1, P_2, P_3 and P_4 are arbitrary number, D^{β} is the fractional derivative for order β and $f(x)$ is unknown function of the crisp variable x, k is an integer. However, assume IVPs (0.2) and each $x > 0$ has a unique fuzzy solution. The VPM calculation is a novel numeric plan created to examine and decipher the arrangement of first and second request dubious IVPs. Computer simulation, biophysics, synthetic science, applied mathematics, geophysics, material science, harmonies, and other domains rely heavily on linear and non-linear fractional equations. This is seeing fractional derivatives must store the record of the parameter under deliberation. The proposed approach targets constructing an answer of a VPM development just as limiting remaining blunder capacities for processing the obscure coefficients of VPM by applying a specific differential administrator without linearly or constraint on the structure. Again, we refer to see numerous qualities to show and reconsider some radical strategies for managing the various problems that occur in ordinary miracles.

2 Preliminaries

The definitions of significance and associated characteristics of the hypothesis are examined in this section.

Definition 2.1. *The fractional component of Riemann-Liouville, f the valued function of the fuzzy number β is considered to include $J_{\alpha}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_0^x \frac{f(\xi)}{(x-\xi)^{1-\beta}} d\xi$, $x > a$ where $\Gamma(\beta)$ is the famous Gamma characteristic.*

Definition 2.2. *The Riemann-Liouville fractional order derivative β of the crisp function f almost everywhere on I exists and can be represented by ${}_{a}^{RL}D^{\beta} f(x) = \frac{1}{\Gamma(m-\beta)} \frac{d^m}{dx^m} \int_0^x f(\xi)(x-\xi)^{m-\beta-1} d\xi$, where $m - 1 \leq \beta < m \in Z^+$.*

Definition 2.3. *The modified Riemann-Liouville fractional order derivative β of the crisp function f almost everywhere on I exists and can be represented by* ${}_{a}^{RL}D^{\beta}f(x) = \frac{1}{\Gamma(m-\beta)}\frac{d^m}{dx^m}\int_0^x (f(\xi) - f(0))(x-\xi)^{m-\beta-1}d\xi$, *where* $m-1 \leq \beta < m$, *and* $m \geq 1$.

Remark 2.1. .

(i) Riemann-Liouville derivative does not satisfy $D^{\alpha}_{\alpha}(1) = 0 \ \ (D^{\alpha}_{\alpha}(1) = 0$ *for the Caputo derivative), if* α *is not a natural number*

(ii) All fractionals do not satisfy the known formula of the derivative of product of the two functions:

$$D^{\alpha}_{\alpha}(fg) = f(D^{\alpha}_{\alpha}(g)) + g(D^{\alpha}_{\alpha}(f))$$

(ii) All fractionals do not satisfy the known formula of the derivative of quotient of the two functions:

$$D^{\alpha}_{\alpha}(\tfrac{f}{g}) = \tfrac{g(D^{\alpha}_{\alpha}(f)) - f(D^{\alpha}_{\alpha}(g))}{g^2}$$

3 Variation of parameters method

We consider the extensive expression to derive the main definition of the VPM

$$L(u) + N(u) + R(u) = f(x), \quad a \leq x \leq b \tag{3}$$

where L, N operators are linear and non-linear. R is a linear differential operator but L has the highest order than R, $f(x)$ is a source term in the given domain $[a, b]$. We have the following equation solution by using the VPM

$$u(x) = \textstyle\sum_{l=0}^{k-1} \frac{p_{l+1}x^l}{l!} + \int_0^x f(x,\alpha)(-N(u)(\alpha) - R(u)(\alpha) + f(\alpha))d\alpha, \tag{4}$$

where k represent the order of given differential equation and C_l where $l = 1, 2, 3$,are unknown. So

$$u(x) = \textstyle\sum_{l=0}^{k-1} \frac{p_{l+1}x^l}{l!} \tag{5}$$

For homogeneous solution which is used by

$$L(u) = 0. \tag{6}$$

Another component obtained from definition (0.2) from VPM is

$$\int_0^x f(x,\alpha)(-N(u)(\alpha) - R(u)(\alpha) + f(\alpha))d\alpha. \tag{7}$$

Where, $f(x,\alpha)$ is a Lagrange multiplier which that eliminates the incremental use of integers in the inverse problem and is dependent on the order of equations. The explore description is based on calculating the function of the variable $f(x,\alpha)$ from either a set of numbers.

$$f(x,\alpha) = \sum_{l=0}^{k-1} \frac{(-1)^{l-1}\alpha^{l-1}x^{k-1}}{(l-1)!(k-1)!} = \frac{(x-\alpha)^{k-1}}{(k-1)!}, \tag{8}$$

The sequence of the specific differential equations differs with k. We have always had the following criteria to explore:

$$\begin{array}{ll} k = 1, & f(x,\alpha) = 1, \\ k = 2, & f(x,\alpha) = (x-\alpha) \\ k = 3, & f(x,\alpha) = \frac{x^2}{2!} + \frac{\alpha^2}{2!} - \alpha x \end{array} \tag{9}$$

As a result, we utilize its investigation to improve, the system to solve equations

$$u_{n+1} = u_0 + \int_0^x f(x,\alpha)(-N(u)(\alpha) - R(u)(\alpha) + f(\alpha))d\alpha. \tag{10}$$

Using initial conditions, we can obtain the initial guess $u_0(x)$. We improve our estimate by using a specific value for the input parameter in each iteration. We are using Reimann-Liouville to solve the fractional Abel differential condition. When we combine VPM with a fractional integral for the arrangement process, the iterative plan for fractional equations is

$$u_{n+1} = u_0 + \frac{1}{\Gamma(\beta)} \int_0^x f(x,\alpha)^{x-\alpha}(-N(u)(\alpha) - R(u)(\alpha) + f(\alpha))d\alpha. \tag{11}$$

4 Numerical Examples

Example 4.1. *We Consider the following fractional Abel problem,*

$$D^\beta f(x) - 3f^3(x) + f(x) = 1, \;\; 0 < \beta \leq 1, \;\; x > 0, \tag{12}$$

with initial condition $f(0) = \frac{1}{3}$ can be found as follows:

$$f(x) = \frac{1}{\sqrt{6e^{2x+3}}}. \tag{13}$$

For the above problem, we create the iterative scheme shown below

$$f_{n+1} = f_0 + \frac{1}{\Gamma(\beta)} \int_0^x (x-\alpha)^{\beta-1}(3f^3(x) - f(x))d\alpha \tag{14}$$

x	Exact value	VPM	Absolute error
0.0	[0.3333]	[0.2916]	[0.0417]
0.1	[0.3111]	[0.3062]	[0.0049]
0.2	[0.2892]	[0.2730]	[0.0162]
0.3	[0.2679]	[0.2531]	[0.0148]
0.4	[0.2472]	[0.2374]	[0.0098]
0.5	[0.2275]	[0.2164]	[0.0111]
0.6	[0.2088]	[0.1943]	[0.0145]
0.7	[0.1912]	[0.1804]	[0.0108]
0.8	[0.1748]	[0.1692]	[0.0056]
0.9	[0.1595]	[0.1439]	[0.0156]
1.0	[0.1453]	[0.1379]	[0.0074]

Table 1: Value of f(x), $\beta = 1$

x	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$
0.0	0.3201	0.3197	0.2932
0.2	0.2817	0.2809	0.2782
0.4	0.2402	0.2397	0.2381
0.6	0.2007	0.1991	0.1962
0.8	0.1731	0.1706	0.1699
1.0	0.1497	0.1399	0.1388

Table 2: Different values of β

Taking initial condition $f(0) = \frac{1}{3}$, the following results for $\beta = 1$ are produced:

$f_1(x) = \frac{1}{3} - \frac{2}{9}x$

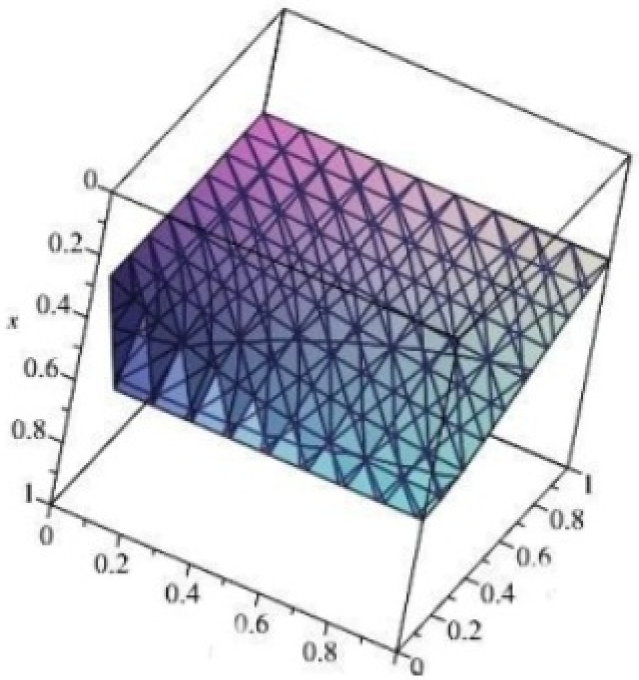
$f_2(x) = \frac{7}{24} - \frac{4}{9}x + \frac{4}{81}x^3 - \frac{2}{243}x^4$

$f_3(x) = \frac{7}{24} - \frac{3049}{4608}x + \frac{5}{96}x^2 + \frac{2}{9}x^3 - \frac{133}{1728}x^4 - \frac{881}{3880}x^5 + \frac{13}{729}x^6 - \frac{2}{1701}x^7$

$- \frac{13}{8748}x^8 + \frac{5}{13122}x^9 + \frac{8}{885735}x^{10} - \frac{32}{1948617}x^{11} + \frac{4}{1594323}x^{12} - \frac{8}{62178597}x^{13}.$

Table 1 show a approximate solution and exact solution for $\beta = 1$. Table 2 shows different values of β. Fig 1 represents the exact and approximate solution.

Figure 1: Value of f(x)



Example 4.2. *We Consider the following fractional Abel problem,*

$$D^{\beta}f(x)+f^{3}(x)-f(x)=1,\ \ 0<\beta\leq 1,\ \ x>0, \tag{15}$$

with initial condition $f(0)=\frac{1}{3}$ *can be found as follows:*

$$f(x)=\frac{e^{t}}{\sqrt{e^{2t}+8}}. \tag{16}$$

For the above problem, we create the iterative scheme shown below

$$f_{n+1}=f_0+\frac{1}{\Gamma(\beta)}\int_0^x(x-\alpha)^{\beta-1}(f(x)-f^{3}(x))d\alpha \tag{17}$$

Taking initial condition $f(0)=\frac{1}{3}$*, the following results for* $\beta=1$ *are produced:*
$f_1(x)=\frac{1}{3}+\frac{8}{27}x$
$f_2(x)=\frac{31}{96}+\frac{16}{27}x+\frac{8}{81}x^2-\frac{64}{2187}x^3-\frac{128}{19683}x^4$

$f_3(x)=\frac{31}{96}+\frac{780193}{884736}x+\frac{1045}{3456}x^2-\frac{11201}{93312}x^3-\frac{38591}{419904}x^4-\frac{2657}{157464}x^5+\frac{784}{177147}x^6+\frac{3776}{1594323}x^7$
$-\frac{2240}{14348907}x^8-\frac{100864}{1162261467}x^9-\frac{851968}{52301766015}x^{10}-\frac{131072}{345191655699}x^{11}+\frac{262144}{847288609443}x^{12}+\frac{2097152}{99132767304831}x^{13}.$

x	Exact value	VPM	Absolute error
0.0	[0.3333]	[0.3229]	[0.0104]
0.1	[0.3639]	[0.3508]	[0.0131]
0.2	[0.3964]	[0.3872]	[0.0092]
0.3	[0.4307]	[0.4139]	[0.0168]
0.4	[0.4665]	[0.4477]	[0.0188]
0.5	[0.5035]	[0.4972]	[0.0063]
0.6	[0.5415]	[0.5128]	[0.0287]
0.7	[0.5799]	[0.5524]	[0.0275]
0.8	[0.6183]	[0.6106]	[0.0077]
0.9	[0.6561]	[0.6392]	[0.0169]
1.0	[0.6929]	[0.6548]	[0.0381]

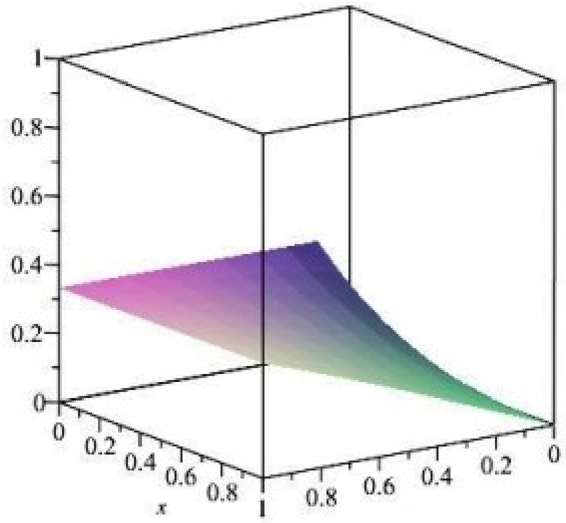
Table 3: Value of f(x), $\beta = 1$

x	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$
0.0	0.3301	0.3299	0.3271
0.2	0.3956	0.3907	0.3882
0.4	0.4641	0.4596	0.4481
0.6	0.5364	0.5209	0.5197
0.8	0.6180	0.6159	0.6132
1.0	0.6877	0.6792	0.6674

Table 4: Different values of β

Table 3 shows a approximate solution and exact solution for $\beta = 1$. Table 4 shows different values of β. Fig 2 represents the exact and approximate solution.

Figure 2: Value of f(x)



5 Conclusions

In this study, we examined the formulation to fractional Abel equation with the Variation of Parameters method. We demonstrate that VPM is a functional, efficient approach for the achievement of empirical and numerical testing for a wide variety of nonlinear fractional equations. We discovered that obtained estimate effects for different beta values interact simultaneously before the first-order derivative is exceeded.

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