# Relation-theoretic contraction principle in metric spaces using multiplicative contraction

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**Abstract**

Alam and Imdad have presented a novel application of the Banach contraction principle on a complete metric spaces with a binary relation. We have extended the concept of binary relation with the multiplicative contraction in a complete metric spaces. We have also included corollary to demonstrate our results.
**Keywords**: fixed point, metric spaces, binary relation, multiplicative contraction
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## 1 Introduction

In many scientific domains, particularly in fixed point theory, the concept of a metric space is extremely useful. This notion has been generalised in numerous directions in recent years, and many notions of a metric-type space have been introduced (b-metric, dislocated space, generalised metric space, quasi-metric space, symmetric space, etc.).The Banach contraction principle's [3]contraction condition has been generalised to numerous forms in the last fifty years. Furthermore, the metric space in the Banach contraction principle has been generalised to a variety of generalised metric spaces. Many authors researched other sorts of fixed point theorems in metric spaces later on, as seen by the and references therein.

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## 2 Preliminaries

We give the required background material needed to prove our results in this part to make our exposition self-contained. In what follows, $N, N_0, Q$ and $R$ denote the sets of positive integers, non-negative integers, rational numbers and real numbers, respectively.
Aftab and Alam [1] proved new relation-theoretic fixed point theorems on metric spaces in 2015, and then inferred comparable findings in metric spaces.
Metric spaces are sets in which there is a defined a notion of '*distance between pair of points*'. The concept of metric spaces was formulated in 1906 by M.Frechet [7] ,though the definition presently in use given by the German mathematician, Felix Hausdorff.

**Definition 2.1.** *Let $M$ be a non empty arbitrary set and $d$ be a real function from $M \times M$ into $R^+$ such that for all $u, v, w \in M$ we have*

1. $d(u, v) \geq 0$,
2. $d(u, v) = 0 \iff u = v$,
3. $d(u, v) = d(v, u)$ *and*
4. $d(u, w) \leq d(u, v) + d(v, w)$,

*Here $(M, d)$ is called a metric in $R$ and $(R, d)$ is a metric space.*

**Example 2.1.** 1. $d(u, v) = |u - v|$ *is a metric space in $R$.*

2. *If $d(u, v)$ defined by*

$$d(u, v) = \{1 \text{ if } u \neq v$$

$$0 \text{ if } u = v\}$$

**Definition 2.2.** *[10] Let $M$ be a nonempty set. A subset $R$ of $M^2$ is called a binary relation on $M$. Notice that for each pair $u, v \in M$, one of the following conditions holds:*

1. $(u, v) \in R$; *which amounts to saying that "$u$ is $R$-related to $v$" or "$u$ relates to $v$ under $R$". Sometimes, we write $uRv$ instead of $(u, v) \in R$ ;*

2. $(u, v) \notin R$; *which means that "$u$ is not $R$-related to $v$" or "$u$ does not relate to $v$ under $R$".*

**Definition 2.3.** *[10] Let $R$ be a binary relation defined on a nonempty set $M$ and $u, v \in M$. We say that $u$ and $v$ are $R$ -comparative if either $(u, v) \in R$ or $(v, u) \in R$. We denote it by $[u, v] \in R$.*

**Definition 2.4.** *[6, 10, 11, 12, 15] A binary relation $R$ defined on a nonempty set $M$ is called*

1. *reflexive if $(u, u) \in R$ for all $u \in M$,*
2. *irreflexive if $(u, u) \notin R$ for all $u \in M$,*
3. *symmetric if $(u, v) \in R$ implies $(v, u) \in R$,*
4. *antisymmetric if $(u, v) \in R$ implies $(v, u) \notin R$,*
5. *transitive if $(u, w) \in R$ and $(w, v) \in R$ implies $(u, v) \in R$,*
6. *complete, connected or dichotomous if $[l, n] \in R$ for all $l, m \in M$,*
7. *weakly complete, weakly connected or trichotomous if $[u, v] \in R$ or $u = v$ for all $u, v \in M$.*
8. *strict order or sharp order if $R$ is irreflexive and transitive,*
9. *near-order if $R$ is antisymmetric and transitive,*
10. *pseudo-order if $R$ is reflexive and antisymmetric,*
11. *quasi-order or preorder if $R$ is reflexive and transitive,*
12. *partial order if $R$ is reflexive, antisymmetric and transitive,*
13. *simple order if $R$ if weakly complete strict order,*
14. *weak order if $R$ is complete preorder,*
15. *total order, linear order or chain if $R$ is complete partial order,*
16. *tolerance if $R$ is reflexive and symmetric,*
17. *equivalence if $R$ is reflexive, symmetric and transitive.*

**Definition 2.5.** [4] *Let $M$ be a nonempty set and $R$ a binary relation on $M$. A sequence $\{u_n\} \subset M$ is called $R$-preserving if*

$$(u_n, u_{n+1}) \in R \quad \forall n \in N_0$$

The notion of $d$- self closeness of a partial order $\preceq$ defined by Turinici [16] is extended to an arbitrary binary relation in the following lines.
Now, we state and prove our main result, which is as follows:

**Theorem 2.1.** *Let $(M, d)$ be a complete metric space, $R$ a binary relation on $M$ and $T$ a self-mapping on $M$. Suppose that the following conditions hold: a) $M(f; R)$ is nonempty,*
*b) $R$ is $f$-closed,*
*c) either $f$ is continuous or $R$ is $p$-self-closed,*
*d) there exists $\lambda \in [0, 1) d(f(u), f(v)) \leq d(u, v)^{\lambda} \quad forall u, v \in M \qquad with (u, v) \in R$*
*Then $f$ has a fixed point. Moreover, if e) $\Upsilon(u, v, R^s)$ is nonempty, for each $u, v \in M$,*
*then $f$ has a unique fixed point.*

**Proof.** Consider a point $u_0 \in M$. Now we define a sequence $\{u_n\}$ of Picard iterates, i.e., $u_n = fu_{n-1}$ for $n = 1, 2, ...$ From the multiplicative contraction property [13] of $f$ for all $n \in N_0$. As $(u_0, fu_0) \in R$, using condition $(b)$, we get

$$(fu_0, f^2u_0), (f^2u_0, f^3u_0), ..., (f^n u_0, f^{n+1}u_0), ... \in R$$

so that

$$(u_n, u_{n+1}) \in R \qquad n \in N_0. \tag{1}$$

Thus the sequence $\{u_n\}$ is $R$-preserving. Applying the contractivity condition $(d)$ to equation (1), we deduce, for all $n \in N_0$, that

$$d(u_{n+1}, u_n) \leq d(u_n, u_{n-1})^{\lambda} \leq d(u_{n-1}, u_{n-2})^{\lambda^2} \leq ... \leq d(u_1, u_0)^{\lambda^n}.$$

which by induction yields that

$$d(u_{n+1}, u_{n+2}) \leq d(u_0, fu_0)^{\lambda^{n+1}} \quad n \in N_0. \tag{2}$$

Using equation (2) and triangular inequality, for all $n \in N_0, p \in N, p \geq 2$, we have

$$\begin{aligned} d(u_{n+1}, u_{n+p}) &\leq d(u_{n+1}, u_{n+2}) + d(u_{n+2}, u_{n+3}) + ... + d(u_{n+p-1}, u_{n+p}) \\ &\leq d(u_1, u_0)^{\lambda^{n+1} + ... + \lambda^p} \\ &\leq d(u_1, u_0)^{\frac{\lambda^p}{1-\lambda}} \end{aligned}$$

This implies that $d(u_n, u_p) \to 1$ as $(n, p \to \infty)$. Hence the sequence $(x_n) = (f^n u_0)$ is multiplicative Cauchy. By the completeness of $M$, there is $z \in M$ such that $u_n \to z$ as $n \to \infty$. Moreover,

$$d(fz, z) \leq d(fu_n, fz).d(fu_n, z) \leq d(u_n, z)^\lambda.d(fu_n, z) \to 1 \text{ as } n \to \infty,$$

which implies $d(fz, z) = 0$. Therefore this says that $z$ is a fixed point of $f$; that is $fz = z$.
Now, if there is another point $y$ such that $fy = y$, then

$$d(z, y) = d(fz, fy) \leq d(z, y)^\lambda.$$

Therefore $d(z, y) = 0$ and $y = z$. This implies that $z$ is the unique fixed point of $f$.
Alternatively, let us assume that $R$ is $d$-self-closed. As $u_n$ is an $R$-preserving sequence and

$$u_n \to^d u,$$

there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with

$$[u_{n_k}, u] \in R \quad k \in N_0$$

Using (d), $[u_{n_k}, u] \in R$ and $u_n \to^d u$, we obtain

$$d(u_{n_k+1}, fu) = d(fu_{n_k}, fu) \leq d(u_{n_k}, u)^\lambda \to 1 \text{ as } k \to \infty$$

so that $u_{n_k+1} \to^d f(u)$. Again, owing to the uniqueness of limit, we get $f(u) = u$ so that $u$ is a fixed point of $f$. To prove uniqueness, take $u, v \in F(f)$, i.e.,

$$f(u) = u \quad and \quad f(v) = v. \tag{3}$$

By assumption $(d)$, there exists a path (say $\{z_0, z_1, z_2, ..., z_k\}$) of some finite length $k$ in $R^s$ from $u$ to $v$ so that

$$z_0 = u, z_k = v, [z_i, z_{i+1}] \in R \quad foreach \quad i(0 \leq i \leq k-1). \tag{4}$$

As $R$ is $f$-closed, we have

$$[f^n z_i, f^n z_{i+1}] \in R \quad foreach \quad i(0 \leq i \leq k-1) \quad and \quad for \quad each \quad n \in N_0. \tag{5}$$

Making use of equations (3),(4),(5),, triangular inequality and assumption $(d)$, we obtain

$$d(u, v) = d(f^n z_0, f^n z_k) \leq \sum_{k-1}^{i=0} d(f^n z_i, f^n z_{i+1})$$

$$\leq \sum_{k-1}^{i=0} d(f^{n-1} z_i, f^{n-1} z_{i+1})^\lambda$$

$$\leq \sum_{k-1}^{i=0} d(f^{n-2} z_i, f^{n-2} z_{i+1})^{\lambda^2}$$

$$\leq ... \leq \sum_{k-1}^{i=0} d(z_i, z_{i+1})^{\lambda^n}$$

$$\to 0 \text{ as } n \to \infty$$

so that $u = v$. Hence $f$ has a unique fixed point.□

**Corolary 2.1.** *Let $(M, d)$ be a complete metric space. For $\epsilon$ with $\epsilon > 1$ and $u_0 \in M$, consider the multiplicative closed ball, $\overline{B}_\epsilon(u_0)$. Suppose the mapping $f : M \to M$ satisfies the contraction condition*

$$d(f(u), f(v)) \leq d(u, v)^\lambda \quad forall u, v \in \overline{B}_\epsilon(u_0)$$

*where $\lambda \in [0, 1)$ is a constant $R$ is a relation such that $d(fu_0, u_0) \leq \epsilon^{1-\lambda}$. Then $f$ has a unique fixed point in $\overline{B}_\epsilon(u_0)$.*

**Corolary 2.2.** *Let $(M, d)$ be a complete metric space. If a mapping $f : M \to M$ satisfies for some positive integer $n$,*

$$d(f^n u, f^n v) \leq d(u, v)^\lambda \quad \textit{for all } u, v \in M,$$

*where $\lambda \in [0, 1)$ is a constant, then $f$ has a unique fixed point in $M$.*

**Theorem 2.2.** *Let $(M, d)$ be a complete metric space, $R$ a binary relation on $M$ and $T$ a self-mapping on $M$. Suppose that the following conditions hold: a) $M(f; R)$ is nonempty,*
*b) $R$ is $f$-closed,*
*c) either $f$ is continuous or $R$ is $p$-self-closed,*
*d) there exists $\lambda \in [0, \frac{1}{2})$ $\quad d(fu, fv) \leq (d(f(u, v).d(fv, u))^\lambda forall u, v \in M \quad with (u, v) \in R$*
*Then $f$ has a fixed point. Moreover, if e) $\Upsilon(u, v, R^s)$ is nonempty, for each $u, v \in M$,*
*then $f$ has a unique fixed point.*

**Proof.** Consider a point $u_0 \in M$. Now we define a sequence $\{u_n\}$ of Picard iterates, i.e., $u_n = fu_{n-1}$ for $n = 1, 2, ...$ From the multiplicative contraction property of $f$ for all $n \in N_0$. As $(u_0, fu_0) \in R$, using condition 2, we get

$$(fu_0, f^2u_0), (f^2u_0, f^3u_0), ..., (f^nu_0, f^{n+1}u_0), ... \in R$$

we have

$$d(u_{n+1}, u_n) = d(fu_n, fu_{n-1}) \leq (d(fu_n, u_n).d(fu_{n-1}, u_{n-1}))^\lambda$$

$$= (d(u_{n+1}, u_n).d(u_n, u_{n-1}))^\lambda$$

Thus we have

$$d(u_{n+1}, u_n) \leq (d(u_n, u_{n-1}))^{\frac{\lambda}{1-\lambda}} = d(u_n, u_{n-1})^h,$$

where $h = \frac{\lambda}{1-\lambda}$. For $n > m$, Using triangular inequality, for all $n \in N_0, m \in N, m \geq m$, we have

$$d(u_n, u_m) \leq d(u_n, u_{n-1}).d(u_{n-1}, u_{n-2})...d(u_{m+1}, u_m)$$

$$\leq d(u_1, u_0)^{h^{n-1}+h^{n-2}+...+h^m} \leq d(u_1, u_0)^{\frac{h^m}{1-h}}$$

This implies $d(u_n, u_m) \to 1$ as $(n, m \to \infty)$. Hence $(u_n)$ is a Cauchy sequence. By the completeness of $M$, there is $z \in M$ such that $u_n \to z$ as $n \to \infty$. Since

$$d(fz, z) \leq d(fu_n, fz).d(fu_n, z)$$

$$\leq (d(fu_n, u_n).d(fz, z))^{\lambda}.d(u_{n+1}, z),$$

we have

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$$d(fz, z) \leq d(fu_n, fz).d(fu_n, z)$$

$$\leq (d(fu_n, u_n).d(fz, z))^{\lambda}.d(u_{n+1}, z),$$

we have

$$d(fz, z) \leq (d(fu_n, u_n)^{\lambda}.d(u_{n+1}, z))^{\frac{1}{1-\lambda}} \to 1 \text{ as } n \to \infty.$$

Hence $d(fz, z) = 0$. This implies $fz = z$. Finally, it is easy to prove that the fixed point of $f$ is unique.
Alternatively, let us assume that $R$ is $d$-self-closed. As $\{u_n\}$ is an $R$-preserving sequence and

$$u_n \to^d u,$$

there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with

$$[u_{n_k}, u] \in R \quad k \in N_0$$

Using $(d)$, $\quad [u_{n_k}, u] \in R$ and $u_{n_k} \to^d u$, we obtain

$$d(u_{n_k+1}, fu) = d(fu_{n_k}, fu) \leq (d(u_{n_k+1}, u).d(u_{n_k}, u))^{\lambda} \to 1 \text{ as } k \to \infty$$

so that $u_{n_k+1} \to^d f(u)$. Again, owing to the uniqueness of limit, we obtain $f(u) = u$, so that $u$ is a fixed point of $f$.

To prove uniqueness, take $u, v \in F(f)$, i.e.,

$$f(u) = u \quad and \quad f(v) = v. \tag{6}$$

By assumption $(e)$, there exists a path (say$\{z_0, z_1, z_2, ..., z_k\}$) of some finite length $k$ in $R^s$ from $u$ to $v$ so that

$$z_0 = u, z_k = v, [z_i, z_{i+1}] \in R \quad for \quad each \quad i(0 \leq i \leq k-1). \tag{7}$$

As $R$ is $f$-closed, we have

$$[f^n z_i, f^n z_{i+1}] \in R \quad for \quad each \quad i(0 \leq i \leq k-1) \quad and \quad for \quad each \quad n \in N_0. \tag{8}$$

Making use of equations (6),(7),(8),, triangular inequality and assumption $(d)$ we obtain

$$d(u, v) = d(f^n z_0, f^n z_k) \leq \sum_{k-1}^{i=0} d(f^n z_i, f^n z_{i+1})$$

$$\leq \sum_{k-1}^{i=0} (d(f^{n-1} z_i, f^{n-1} z_{i+1}).d(f^{n-1} z_{i+1}, f^{n-1} z_i))^{\lambda}$$

$$\leq \sum_{k-1}^{i=0} (d(f^{n-2} z_i, f^{n-2} z_{i+1}).d(f^{n-2} z_{i+1}, f^{n-2} z_i))^{\lambda^2}$$

$$\leq ... \leq \sum_{k-1}^{i=0} (d(z_i, z_{i+1}).d(z_{i+1}, z_i))^{\lambda^n}$$

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$$(fu_0, f^2u_0), (f^2u_0, f^3u_0), ..., (f^nu_0, f^{n+1}u_0), ... \in R$$

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$$d(u_{n+1}, u_n) = d(fu_n, fu_{n-1}) \leq (d(fu_n, u_n).d(fu_{n-1}, u_{n-1}))^{\lambda}$$

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Thus we have

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$$d(u_n, u_m) \leq d(u_n, u_{n-1}).d(u_{n-1}, u_{n-2})...d(u_{m+1}, u_m)$$

$$\leq d(u_1, u_0)^{h^{n-1}+h^{n-2}+...+h^m} \leq d(u_1, u_0)^{\frac{h^m}{1-h}}$$

This implies $d(u_n, u_m) \to 1$ as $(n, m \to \infty)$. Hence $(u_n)$ is a Cauchy sequence. By the completeness of $M$, there is $z \in M$ such that $u_n \to z$ as $n \to \infty$. Since

$$d(fz, z) \leq d(fu_n, fz).d(fu_n, z)$$

$$\leq (d(fu_n, u_n).d(fz, z))^{\lambda}.d(u_{n+1}, z),$$

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Using $(d)$, $[u_{n_k}, u] \in R$ and $u_{n_k} \to^d u$, we obtain

$$d(u_{n_k+1}, fu) = d(fu_{n_k}, fu) \leq (d(u_{n_k+1}, u).d(u_{n_k}, u))^{\lambda} \rightarrow 1 \text{ as } k \rightarrow \infty$$

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To prove uniqueness, take $u, v \in F(f)$, i.e.,

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By assumption $(e)$, there exists a path (say$\{z_0, z_1, z_2, ..., z_k\}$) of some finite length $k$ in $R^s$ from $u$ to $v$ so that

$$z_0 = u, z_k = v, [z_i, z_{i+1}] \in R \quad foreach \quad i(0 \leq i \leq k-1). \tag{10}$$

As $R$ is $f$-closed, we have

$$[f^n z_i, f^n z_{i+1}] \in R \quad foreach \quad i(0 \leq i \leq k-1) \quad and \quad for \quad each \quad n \in N_0. \tag{11}$$

Making use of equations (9),(10),(11), triangular inequality and assumption $(d)$, we obtain

$$d(u, v) = d(f^n z_0, f^n z_k) \leq \sum_{k-1}^{i=0} d(f^n z_i, f^n z_{i+1})$$

$$\leq \sum_{k-1}^{i=0} (d(f^{n-1} z_i, f^{n-1} z_{i+1}).d(f^{n-1} z_{i+1}, f^{n-1} z_i))^{\lambda}$$

$$\leq \sum_{k-1}^{i=0} (d(f^{n-2} z_i, f^{n-2} z_{i+1}).d(f^{n-2} z_{i+1}, f^{n-2} z_i))^{\lambda^2}$$

$$\leq ... \leq \sum_{k-1}^{i=0} (d(z_i, z_{i+1}).d(z_{i+1}, z_i))^{\lambda^n}$$

$$\rightarrow 0 \text{ as } n \rightarrow \infty$$

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