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# Some topological properties of revised fuzzy cone metric spaces

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**Abstract**

In this paper, we introduced Revised fuzzy cone Metric space with its topological properties. Likewise A necessary and sufficient condition for a Revised fuzzy cone metric space to be precompact is given. We additionally show that each distinct Revised fuzzy cone metric space is second countable and that a subspace of a separable Revised fuzzy cone metric space is separable.
**Keywords**: Revised fuzzy metric space; Revised fuzzy cone metric space; separable; second countable.
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# 1 Introduction

After Zadeh [1965] introduced the idea of fuzzy sets, several authors have introduced and studied many notions of metric indistinctness [Huang and Zhang, 2007, Kramosil and Michalek, 1975, Ahmad and Mesiarová-Zemánková, 2007, Navara, 2007, Muraliraj and Thangathamizh, 2021a,b,d, Oner and Tanay, 2015, Grigorenko et al., 2020] and metric cone indistinctness. By modifying the idea of metric indistinctness introduced by George and Veeramani [1994], Zadeh [1965] studied the notion of fuzzy cone metric areas. Especially, they evidenced that every fuzzy cone topological space generates a Hausdorf first-countable topology. Here we tend to study additional topological properties of these areas whose fuzzy metric version are usually found in George and Veeramani [1994], Ghareeb and Al-Omeri [2018], Grabiec [1988], Gregori et al. [2011].

Sostak [2018] additionally represented the idea of "George–Veeramani Fuzzy Metrics Revised". Presently Olga Grigorenko, Juan Jose Minana, Alexander Sostak, Muraliraj and Thangathamizh [2021c] have introduced "On $t$-conorm primarily based Fuzzy (Pseudo) metrics". Recently Muraliraj and Thangathamizh [2021a,c,d] proved various fixed point theorems in revised fuzzy metric spaces. Muraliraj and Thangathamizh [2021b] introduce the concept of Revised fuzzy modular metric space. Moreover, we tend to prove that a Revised fuzzy cone topological space is precompact if and providing each sequence in it's a Cauchy subsequence. Further, we tend to show that $X_1 \times X_2$ may be a complete Revised fuzzy cone topological space if and providing $X_1$ and $X_2$ are complete Revised fuzzy cone metric areas. Finally it's tried that each divisible Revised fuzzy cone topological space is second calculable and a mathematical space of a separable Revised fuzzy cone topological space is separable.

# 2 Preliminaries

**Definition 2.1** (Gregori et al. [2011]). *Let $E$ be a real Banach space, $\theta$ the zero of $E$ and $P$ a subset of $E$. Then $P$ is called a cone if and only if*

*(i) $P$ is closed, nonempty, and $P \neq \{\theta\}$,*

*(ii) if $ab \in R, ab \geq 0$ and $xy \in P$, then $ax + by \in P$,*

*(iii) if both $x \in P$ and $-x \in P$, then $x = \theta$.*

Given a cone $P$, a partial ordering $\precsim$ on $E$ with respect to $P$ is defined by $x \precsim y$ if only if $y - x \in P$. The notation $x \prec y$ will stand for $x \precsim y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int}(P)$. Throughout this paper, we assume that all the cones have nonempty interiors.

There are two kinds of cones: normal and non-normal ones. A cone $P$ is called normal if there exists a constant $K \geq 1$ such that for all $t, s \in E$, $\theta \precsim t \precsim s$ implies $\|t\| \leq K \|s\|$, and the least positive number $K$ having this property is called normal constant of $P$ Gregori et al. [2011]. It is clear that $K \geq 1$.

**Definition 2.2** (Sostak [2018])**.** *A binary operation $\oplus : [0,1] \times [0,1] \to [0,1]$ is a $t$-conorm if it satisfies the following conditions:*

*(i) $\oplus$ is associative andcommutative,*

*(ii) $\oplus$ is continuous,*

*(iii) $a \oplus 0 = a$ for all $a \in [0,1]$,*

*(iv) $a \oplus b \leq c \oplus d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.*

## 2.1 Examples Sostak and Öner [2020]

(i) Lukasievicz $t$-conorm: $a \oplus b = \max\{a, b\}$;

(ii) Product $t$-conorm: $a \oplus b = a + b - ab$;

(iii) Minimum $t$-conorm: $a \oplus b = \min(a + b, 1)$.

**Definition 2.3** (Oner and Tanay [2015])**.** *A 3-tuple $(U, M_0, *)$ is called a fuzzy cone metric space (FCM space) if $C$ is a cone of $E$, $U$ is an arbitrary set, $(*)$ is a continuous $t$-norm, and $M_0$ is a fuzzy set on $U^2 \times \mathrm{int}(P)$ satisfying the following conditions:*

*(i) $M_0(\lambda_1, \lambda_2, t) > 0$ and $M_0(\lambda_1, \lambda_2, t) = 1 \Leftrightarrow \lambda_1 = \lambda_2$,*

*(ii) $M_0(\lambda_1, \lambda_2, t) = M_0(\lambda_2, \lambda_1, t)$*

*(iii) $M_0(\lambda_1, \lambda_2, t) * M_0(\lambda_2, \lambda_3, s) \leq M_0(\lambda_1, \lambda_3, t + s)$,*

*(iv) $M_0(\lambda_1, \lambda_2, ) : \mathrm{int}(P) \to [0,1]$ is continuous $\forall \lambda_1, \lambda_2, \lambda_3 \in U$ and $t, s \in int(p)$.*

**Definition 2.4** (Sostak [2018])**.** *A Revised fuzzy metric space is an ordered triple $(X, \mu, \oplus)$ such that $X$ is a non empty set, $\oplus$ is a continuous $t$-conorm and $\mu$ is a Revised fuzzy set on $\mu : X^2 \times R^+ \to [0,1]$ satisfies the following conditions:*

*(RFM1) $\mu(x, y, t) < 1$;*

*(RFM2) $\mu(x, y, t) = 0$ if and only if $x = y$;*

*(RFM3) $\mu(x, y, t) = \mu(y, x, t)$;*

*(RFM4) $\mu(x, z, t + s) \leq \mu(x, y, t) \oplus \mu(y, z, s)$;*

*(RFM5) $\mu(x, y, -) : (0, \infty) \to [0, 1)$ is continuous $\forall x, y, z \in X$ and $t, s \in R^+$. Then $\mu$ is called a Revised fuzzy metric on $X$.*

# 3 Main Results

**Definition 3.1.** *A Revised fuzzy cone metric space is an 3-triple* $(X, \mu, \oplus)$ *such that* $P$ *is a cone of* $E$*,* $X$ *is a non empty set,* $\oplus$ *is a continuous* $t$*-conorm and* $\mu$ *is a Revised fuzzy set on* $X^2 \times \operatorname{int}(P)$ *satisfies the following conditions,* $\forall x, y, z \in X$ *and* $s, t \in int(P)$ *(that is* $s \gg \theta, t \gg \theta$*),*

*(RFCM 1)* $\mu(x, y, t) < 1$ ,

*(RFCM 2)* $\mu(x, y, t) = 0$ *if and only if* $x = y$*,*

*(RFCM 3)* $\mu(x, y, t) = \mu(y, x, t)$*,*

*(RFCM 4)* $\mu(x, z, t + s) \leq \mu(x, y, t) \oplus \mu(y, z, s)$*,*

*(RFCM 5)* $\mu(x, y, -) : \operatorname{int}(P) \to [0, 1]$ *is continuous.*

*Then* $\mu$ *is called a Revised fuzzy cone metric on* $X$*.*

If $(X, \mu, \oplus)$ is a Revised fuzzy cone metric space, we will say that $\mu$ is a Revised fuzzy cone metric on $X$.

Every revised fuzzy cone metric space $(X, \mu, \oplus)$ induces a Hausdorff first-countable topology $\tau f_c$ on $X$ which has as a base the family of sets of the form $\{B(x, r, t) : x \in X; 0 < r < 1, t \gg \theta\}$,
where $\{B(x, r, t) : y \in X; \mu(x, y, t) < r\}$ for every $r$ with $0 < r < 1$ and $t \gg \theta$.

A Revised fuzzy cone metric space $(X, \mu, \oplus)$ is called complete if every Cauchy sequence in it is convergent, where a sequence $\{x_n\}$ is said to be a Cauchy sequence if for any $\varepsilon \in (0, 1)$ and any $t \gg \theta$ there exists a natural number $n_0$ such that $\mu(x_n, x_m, t) < \varepsilon$ for all $n, m \geq n_0$, and a sequence $\{x_n\}$ is said to converge to $x$ if for any $t \gg \theta$ and any $r \in (0, 1)$ there exists a natural number $n_0$ such that $\mu(x_n, x, t) < r$ for all $n \geq n_0$. A sequence $\{x_n\}$ converges to $x$ if and only if $\mu(x_n, x, t) \to 0$ for each $t \gg \theta$.

**Definition 3.2.** *Let* $(X, \mu, \oplus)$ *be a Revised fuzzy cone metric space. For* $t \gg \theta$*, the closed ball* $B[x, r, t]$ *with center* $x$ *and radius* $r \in (0, 1)$ *is defined by*

$$B[x, r, t] = \{y \in X; \mu(x, y, t) < r\}.$$

**Lemma 3.3.** *Every closed ball in a Revised fuzzy cone metric space* $(X, \mu, \oplus)$ *is a closed set.*

*Proof.* Let $y \in \overline{B[x, r, t]}$. Since $X$ is first countable, there exits a sequence $\{y_n\}$ in $B[x, r, t]$ converging to $y$. Therefore $\mu(y_n, y, t)$ converges to 0 for all $t \gg \theta$. For a given $\epsilon \gg 0$, we have, $\mu(x, y, t + \epsilon) \leq \mu(x, y_n, t) \oplus \mu(y_n, y, \epsilon)$

Hence, $\mu(x, y, t + \epsilon) \leq \mu(x, y_n, t) \oplus \mu(y_n, y, \epsilon) \leq 0 \oplus 0 = 0$.

(If $\mu(x, y_n, t)$ is bounded, then the sequence $y_n$ has a subsequence, which we again denote by $y_n$, for which $\mu(x, y_n, t)$ exists.) In particular for $n \in N$, take $\epsilon = \frac{t}{n}$. Then, $\mu\left(x, y, t + \frac{t}{n}\right) < r$. Hence, $\mu(xyt) \leq \mu\left(x, y, t + \frac{t}{n}\right) < r$. Thus $y \in B[x, r, t]$. Therefore $B[x, r, t]$ is a closed set. □

**Definition 3.4.** *A Revised fuzzy cone metric space $(X, \mu, \oplus)$ is called precompact if for each r, with $0 < r < 1$, and each t $\gg \theta$, there is a finite subset $A$ of $X$, such that $X = \bigcup_{a \in A} B(a, r, t)$. In this case, we say that $\mu$ is a precompact Revised fuzzy cone metric on $X$.*

**Lemma 3.5.** *A Revised fuzzy cone metric space is precompact if and only if every sequence has a Cauchy subsequence.*

*Proof.* Suppose that $(X, \mu, \oplus)$ is a precompact Revised fuzzy cone metric space. Let $x_n$ be a sequence in $X$. For each $m \in N$ there is a finite subset $A_m$ of $X$ such that $X = \bigcup_{a \in A} B\left(a, \frac{a}{m}, \frac{t_0}{m\|t_0\|}\right)$ where $t_0, t \gg \theta$ is a constant. Hence, for $m = 1$, there exists an $a_1 \in A_1$ and a subsequence $x_{1(n)}$ of $x_n$ such that $x_{1(n)} \in B\left(a_1, 1, \frac{t_0}{m\|t_0\|}\right)$ for every $n \in N$. Similarly, there exist an $a_2 \in A_2$ and a subsequence $\left\{x_{2(n)}\right\}$ of $x_{1(n)}$ such that $x_{2(n)} \in B\left(a_2, \frac{1}{2}, \frac{t_0}{m\|t_0\|}\right)$ for every $n \in N$. By continuing this process, we get that for $m \in N$, $m > 1$, there is an $a_m \in A_m$ and a subsequence $\left\{x_{m(n)}\right\}$ of $x_{m-1(n)}$ such that $x_{m(n)} \in B\left(a_m, \frac{1}{m}, \frac{t_0}{m\|t_0\|}\right)$ for every $n \in N$. Now, consider the subsequence $x_{n(n)}$ of $x_n$. Given $r$ with $0 < r < 1$ and $t \gg \theta$ there is an $n_0 \in N$ such that $\frac{1}{n_0} \oplus \frac{1}{n_0} < r$ and $\frac{2t_0}{n_0\|t_0\|} \ll t$. Then, for every $km \geq n_0$, we have

$$\begin{aligned}
\mu\left(x_{k(k)}, x_{m(m)}, t\right) &\leq \mu\left(x_{k(k)}, x_{m(m)}, \frac{t_0}{n_0\|t_0\|}\right) \\
&\leq \mu\left(x_{k(k)}, a_{n_0}, \frac{t_0}{n_0\|t_0\|}\right) \oplus \mu\left(a_{n_0}, x_{m(m)}, \frac{t_0}{n_0\|t_0\|}\right) \\
&\leq \frac{1}{n_0} \oplus \frac{1}{n_0} < r
\end{aligned}$$

Hence $\left(x_{n(n)}\right)$ is a Cauchy sequence in $(X, \mu, \oplus)$.

Conversely, suppose that $(X, \mu, \oplus)$ is a nonprecompact Revised fuzzy cone metric space. Then there exist an $r$ with $0 < r < 1$ and $t \gg \theta$ such that for each finite subset A of $X$, we have $X \neq \bigcup_{a \in A} B(a, r, t)$ fix $x_1 \in X$. There is $x_2 \in X - B(x_1 r t)$. Moreover, there is an $x_3 \in X - \bigcup_{k=1}^{2} B(x_k, r, t)$. By continuing this process, we construct a sequence $x_n$ of distinct points in $X$ such that $x_{n+1} \notin X - \bigcup_{k=1}^{n} B(x_k, r, t)$ for every $n \in N$. Therefore $x_n$ has no Cauchy subsequence. This completes the proof. □

**Lemma 3.6.** *Let $(X, \mu, \oplus)$ be a Revised fuzzy cone metric space. If a Cauchy sequence clusters around a point $x \in X$, then the sequence converges to $x$.*

*Proof.* Let $x_n$ be a Cauchy sequence in $(X, \mu, \oplus)$ having a cluster point $x \in X$. Then, there is a subsequence $\left\{x_{k(n)}\right\}$ of $x_n$ that converges to $x$ with respect to $\tau f_c$. Thus, given $r$ with $0 < r < 1$ and $t \gg \theta$, there is an $n_0 \in N$ such that for each $n \geq n_0$, $\mu\left(x, x_{k(n)}, \frac{t}{2}\right) < s$ where $s > 0$ satisfies $s \oplus s < r$. On the other hand, there is $n_0 \geq k(n_0)$ such that for each $nm \geq n_1$, we have $\mu\left(x_n, x_m, \frac{t}{2}\right) < s$. Therefore, for each $n \geq n_1$, we have
$\mu\left(x, x_n, t\right) \leq \mu\left(x, x_{k(n)}, \frac{t}{2}\right) \oplus \mu\left(x_{k(n)}, x, \frac{t}{2}\right) \leq s \oplus s < r$.

We conclude that the Cauchy sequence $x_n$ converges to $x$. □

**Proposition 3.7.** *Let $(X_1, \mu_1, \oplus)$ and $(X_2, \mu_2, \oplus)$ be Revised fuzzy cone metric spaces. For $\left(x_1, x_2\right), \left(y_1, y_2\right) \in X_1, X_2$, let $\mu\left(\left(x_1, x_2\right), \left(y_1, y_2\right), t\right) = \mu_1\left(x_1, y_1, t\right) \oplus \mu_2\left(x_2, y_2, t\right)$, Then $\mu$ is a Revised fuzzy cone metric on $X_1 \times X_2$.*

*Proof.* **RFCM 1:** Since $\mu_1\left(x_1, y_1, t\right) < 1$ and $\mu_2\left(x_2, y_2, t\right) < 1$, this implies that $\mu_1\left(x_1, y_1, t\right) \oplus \mu_2\left(x_2, y_2, t\right) < 1$. Therefore, $\mu\left(\left(x_1, x_2\right), \left(y_1, y_2\right), t\right) < 1$.
**RFCM 2:** Suppose that for all $t \gg \theta$, $\left(x_1, y_1, t\right) = \left(x_2, y_2, t\right)$. This implies that $x_1 = y_1$ and $x_2 = y_2$ for all $t \gg \theta$. Hence, $\mu_1\left(x_1, y_1, t\right) = 0$ and $\mu_2\left(x_2, y_2, t\right) = 0$.

It follows that, $\mu\left(\left(x_1, x_2\right), \left(y_1, y_2\right), t\right) = 0$.

Conversely, suppose that $\mu\left(\left(x_1, x_2\right), \left(y_1, y_2\right), t\right) = 0$.
This implies that $\mu_1\left(x_1, y_1, t\right) \oplus \mu_2\left(x_2, y_2, t\right) = 0$.

Since, $0 < \mu_1\left(x_1, y_1, t\right) < 1$ and $0 < \mu_2\left(x_2, y_2, t\right) < 1$. It follows that, $\mu_1\left(x_1, y_1, t\right) = 0$ and $\mu_2\left(x_2, y_2, t\right) = 0$. Thus $x_1 = y_1$ and $x_2 = y_2$. Therefore $\left(x_1, x_2\right) = \left(y_1, y_2\right)$.
**RFCM 3:** To prove that $\mu\left(\left(x_1, x_2\right), \left(y_1, y_2\right), t\right) = \mu\left(\left(y_1, y_2\right), \left(x_1, x_2\right), t\right)$ we observe that $\mu_1\left(x_1, y_1, t\right) = \mu_1\left(y_1, x_1, t\right)$ and $\mu_2\left(x_2, y_2, t\right) = \mu_2\left(y_2, x_2, t\right)$. It follows that for all $\left(x_1, x_2\right)\left(y_1, y_2\right) \in X_1 \times X_2$ and $t \gg \theta$,

$$\mu\left(\left(x_1, x_2\right), \left(y_1, y_2\right), t\right) = \mu\left(\left(y_1, y_2\right), \left(x_1, x_2\right), t\right)$$

**RFCM 4:** Since $(X_1, \mu_1, \oplus)$ and $(X_2, \mu_2, \oplus)$ are Revised fuzzy cone metric spaces, we have that, $\mu_1\left(x_1, z_1, t+s\right) \leq \mu_1\left(x_1, y_1, t\right) \oplus \mu_1\left(y_1, z_1, s\right)$ and
$\mu_2\left(x_2, z_2, t+s\right) \leq \mu_2\left(x_2, y_2, t\right) \oplus \mu_2\left(y_2, z_2, s\right)$, for all
$\left(x_1, x_2\right)\left(y_1, y_2\right)\left(z_1, z_2\right) \in X_1 \times X_2$ and $t, s \gg \theta$. Therefore,

$$\begin{aligned}
&\mu\left(\left(x_1, x_2\right), \left(z_1, z_2\right), t+s\right) \\
&= \mu_1\left(x_1, z_1, t+s\right) \oplus \mu_2\left(x_2, z_2, t+s\right) \\
&\leq \mu_1\left(x_1, y_1, t\right) \oplus \mu_1\left(y_1, z_1, s\right) \oplus \mu_2\left(x_2, y_2, t\right) \oplus \mu_2\left(y_2, z_2, s\right) \\
&\leq \mu_1\left(x_1, y_1, t\right) \oplus \mu_2\left(x_2, y_2, t\right) \oplus \mu_1\left(y_1, z_1, s\right) \oplus \mu_2\left(y_2, z_2, s\right) \\
&\leq \mu_1\left(x_1, y_1, t\right) \oplus \mu_2\left(x_2, y_2, t\right) \oplus \mu_1\left(y_1, z_1, s\right) \oplus \mu_2\left(y_2, z_2, s\right) \\
&\leq \mu\left(\left(x_1, x_2\right), \left(y_1, y_2\right), t\right) \oplus \mu\left(\left(y_1, y_2\right), \left(z_1, z_2\right), t\right)
\end{aligned}$$

**RFCM 5:** Note that $\mu_1(x_1, y_1, t)$ and $\mu_2(x_2, y_2, t)$ are continuous with respect to $t$ and $\oplus$ is continuous too. It follows that,
$\mu((x_1, x_2), (y_1, y_2), t) = \mu_1(x_1, y_1, t) \oplus \mu_2(x_2, y_2, t)$ is also continuous. □

**Proposition 3.8.** *Let $(X_1, \mu_1, \oplus)$ and $(X_2, \mu_2, \oplus)$ be Revised fuzzy cone metric spaces. We define, $\mu((x_1, x_2), (y_1, y_2), t) = \mu_1(x_1, y_1, t) \oplus \mu_2(x_2, y_2, t)$. Then $\mu$ is a complete Revised fuzzy cone metric on $X_1 \times X_2$ if and only if $(X_1, \mu_1, \oplus)$ and $(X_2, \mu_2, \oplus)$ are complete.*

**Corollary 3.9.** *Every separable Revised fuzzy cone metric space is second countable.*

*Proof.* Let $(X, \mu, \oplus)$ be the given separable Revised fuzzy cone metric space. Let $A = \{a_n : n \in N\}$ be a countable dense subset of $X$. Consider

$$B = \left\{ \left( a_j, \frac{1}{k}, \frac{t_1}{k\|t_1\|} \right) : j, k \in N \right\}$$

where $t_1 \gg \theta$ is constant. Then $B$ is countable. We claim that $B$ is a base for the family of all open sets in $X$. Let $G$ be an open set in $X$. Let $x \in G$ then there exists $r$ with $0 < r < 1$ and $t \gg \theta$ such that $B(x, rt) \subset G$.

Since $r \in (0, 1)$, we can find an $s \in (0, 1)$ such that $s \oplus s < r$. Choose $m \in N$ such that $\frac{1}{m} < s$ and $\frac{t_1}{m\|t_1\|} \ll \frac{t}{2}$. Since A is dense in $X$, there exists an $a_j \in A$ such that $a_j \in B\left(x, \frac{1}{m}, \frac{t_1}{m\|t_1\|}\right)$. Now if $y \in B\left(a_j, \frac{1}{m}, \frac{t_1}{m\|t_1\|}\right)$, then

$$\begin{aligned}
\mu(x, y, t) &\le \mu\left(x, a_j, \frac{t}{2}\right) \oplus \mu\left(y, a_j, \frac{t}{2}\right) \\
&\le \mu\left(x, a_j, \frac{t_1}{k\|t_1\|}\right) \oplus \mu\left(x, a_j, \frac{t_1}{k\|t_1\|}\right) \\
&\le \frac{1}{m} \oplus \frac{1}{m} \\
&\le s \oplus s < r < r.
\end{aligned}$$

Thus $y \in B(x, r, t)$ and hence $B$ is a basis. □

**Proposition 3.10.** *A subspace of a separable Revised fuzzy cone metric space is separable.*

*Proof.* Let $X$ be a separable Revised fuzzy cone metric space and $Y$ a subspace of $X$. Let $A = \{x_n : n \in N\}$ be a countable dense subset of $X$. For arbitrary but fixed $nk \in N$, if there are points $x \in X$ such that $\mu\left(x_n, x, \frac{t_1}{k\|t_1\|}\right) < \frac{1}{k}$ where $t_1 \gg \theta$ is constant, choose one of them and denote it by $x_{n_k}$.

Let $B = \{x_{n_k} : n, k \in N\}$ then $B$ is countable. Now we claim that $Y \subset \bar{B}$. Let $x \in Y$. Given $r$ with $0 < r < 1$ and $t \gg \theta$ we can find $k \in N$ such that $\frac{1}{k} \oplus \frac{1}{k} < r$ and $\frac{t_1}{k\|t_1\|} \ll \frac{t}{2}$.

Since $A$ is dense in $X$, there exists an $m \in N$ such that $\mu\left(x_m, y, \frac{t_1}{k\|t_1\|}\right) < \frac{1}{k}$. But by definition of $B$, there exists an $x_{m_k}$ such that $\mu\left(x_{m_k}, x_m, \frac{t_1}{k\|t_1\|}\right) < \frac{1}{k}$. Now

$$\begin{aligned}\mu\left(x_{m_k}, y, \frac{t_1}{k\,\|t_1\|}\right) &\leq \mu\left(x_{m_k}, x_m, \frac{t}{2}\right) \oplus \mu\left(x_m, y, \frac{t}{2}\right) \\ &\leq \mu\left(x_{m_k}, x_m, \frac{t_1}{k\,\|t_1\|}\right) \oplus \mu\left(x_m, y, \frac{t_1}{k\,\|t_1\|}\right) \\ &\leq \frac{1}{k} \oplus \frac{1}{k} < r.\end{aligned}$$

Thus $Y \subset \bar{B}$ and hence $Y$ is separable. □

**Corollary 3.11.** *Let $(X, \mu, \oplus)$ be a Revised fuzzy cone metric space. Then $(X, \tau f_c)$ is Hausdorff.*

**Corollary 3.12.** *Let $(X, \mu, \oplus)$ be a Revised fuzzy cone metric space. Define $\tau f_c = A \subset X : x \in A$ if and only if there exist $r \in (0, 1)$, and $t \gg \theta$ such that $B(x, r, tt) \subset A$, then $\tau f_c$ is a topology on $X$.*

**Corollary 3.13.** *In a Revised fuzzy cone metric space, every compact set is closed and RFC-bounded.*

# 4 Conclusion

In this paper we proved a necessary and sufficient condition for a revised fuzzy cone metric space to be precompact. We also show that every separable revised fuzzy cone metric space is second countable and that a subspace of a separable revised fuzzy cone metric space is separable.

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