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Vague Positive Implicative and Associative *W*- Implicative Ideals of Lattice Wajsberg Algebras

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Abstract

In the present paper, we introduce the notions of vague positive implicative, and vague associative *W*-implicative ideals of lattice Wajsberg algebra. Further, we investigate some relevant properties. Moreover, we obtain some relationship between the vague associative *W*-implicative ideal, and the vague *W*-implicative ideal.

Keywords: Wajsberg algebra; Lattice Wajsberg algebra; *W*-implicative ideal; Vague set; Vague *W*-implicative ideal; Vague positive implicative *W*-implicative ideal; Vague associative *W*-implicative ideal.

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1. Introduction

The concept of Wajsberg algebra was first proposed by Mordchaj Wajsberg[10] in 1935, and analysed by Font, Rodriguez and Torrens[4] in 1984. Zadeh introduced the notion of fuzzy set in 1965. Fuzzy logic has been applied to many fields, from control theory to artificial intelligence. In 1993, the idea of vague set was introduced by Gau and Buehrer[5]. Vague set as an extension of fuzzy sets, the idea of vague set is that the membership of every element can be divided into two aspects including supporting and opposing. It is the new extension not only provides a significant addition to existing theories for handling uncertainties, but it leads to potential areas of further field research and pertinent applications. The authors [9], introduced the notions of vague W-implicative ideals, vague implicative W-implicative ideals of lattice Wajsberg algebra, and investigated some properties.

In this paper, we introduce the definitions of the vague positive implicative W-implicative ideal, and the vague associative W-implicative ideal of lattice Wajsberg algebra. Also, we discuss the relationship between the vague positive implicative W-implicative ideal, and the vague W-implicative ideal.

2. Preliminaries

In this section, we recall some basic definitions and results that are helpful in developing our main results.

Definition 2.1[4] Let $(\mathbb{W}, \rightarrow, *, 1)$ be an algebra with a binary operation "→" and a quasi complement "$*$". Then it is called Wajsberg algebra, if the following axioms are satisfied for all $x, y, z \in \mathbb{W}$,

i $1 \rightarrow x = x$

ii $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$

iii $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

iv $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1.$

Proposition 2.2[4] A Wajsberg algebra $(\mathbb{W}, \rightarrow, *, 1)$is satisfied the following axioms for all $x, y, z \in \mathbb{W}$,

i $x \rightarrow x = 1$

ii If $(x \rightarrow y) = (y \rightarrow x) = 1$ then $x = y$

iii $x \rightarrow 1 = 1$

iv $(x \rightarrow (y \rightarrow x)) = 1$

v If $(x \rightarrow y) = (y \rightarrow z) = 1$ then $x \rightarrow z = 1$

vi $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = 1$

vii $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$

viii $x \rightarrow 0 = x \rightarrow 1^* = x^*$

ix $(x^*)^* = x$

x $(x^* \to y^*) = y \to x$.

Definition 2.3[4] A Wajsberg algebra $(\mathbb{W}, \to, *, 1)$ is called a lattice Wajsberg algebra, if the following conditions are satisfied for all $x, y \in \mathbb{W}$,

i The partial ordering " $\leq$ "on a Wajsberg algebra $\mathbb{W}$, such that $x \leq y$ if and only if $x \to y = 1$

ii $(x \vee y) = (x \to y) \to y$

iii $(x \wedge y) = ((x^* \to y^*) \to y^*)^*$.

Thus, $(\mathbb{W}, \vee, \wedge, *, 0, 1)$ is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Proposition 2.4[4] Let $(\mathbb{W}, \to, *, 1)$ be lattice Wajsberg algebra. Then the following axioms hold for all $x, y, z \in \mathbb{W}$,

i If $x \leq y$ then $x \to z \geq y \to z$ and $z \to x \leq z \to y$

ii $x \leq y \to z$ if and only if $y \leq x \to z$

iii $(x \vee y)^* = (x^* \wedge y^*)$

iv $(x \wedge y)^* = (x^* \vee y^*)$

v $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

vi $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

vii $(x \to y) \vee (y \to x) = 1$

viii $x \to (y \vee z) = (x \to y) \vee (x \to z)$

ix $(x \wedge y) \to z = (x \to z) \vee (y \to z)$

x $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

xi $(x \wedge y) \to z = (x \to y) \wedge (x \to z)$.

Definition 2.5[6] Let$(\mathbb{W}, \to, *, 1)$be a lattice Wajsberg algebra. Then it is called lattice *H*-Wajsberg algebra, if it satisfied $(x \vee y) \vee (((x \wedge y) \to z) = 1$ for all $x, y, z \in \mathbb{W}$.

Note. In a lattice *H*-Wajsberg algebra $\mathbb{W}$, the following hold:

i $x \to (x \to y) = (x \to y)$

ii $x \to (y \to z) = (x \to y) \to (x \to z)$for all $x, y, z \in \mathbb{W}$.

Definition 2.6[6] Let L be a lattice. An ideal I of L is a non-empty subset of L is called a lattice ideal, if the following axioms are satisfied for all $x, y \in I$,

i $x \in I$and$y \leq x$ imply $y \in I$

ii $x, y \in I$implies $x \vee y \in I$.

Definition 2.7[6] Let $(\mathbb{W}, \rightarrow, *, 1)$ be a lattice Wajsberg algebra. Let I be a non-empty subset of $\mathbb{W}$. Then I is called a W-implicative ideal of $\mathbb{W}$, if the following axioms are satisfied for all $x, y \epsilon \mathbb{W}$,

i $0 \in I$

ii $(x \rightarrow y)^* \in I$ and $y \in I$ imply $x \in I$.

Definition 2.8[5] A vague set A in the universal of discourse $\mathbb{W}$ is characterized by two membership functions given by:

i A truth membership function $t_A: X \rightarrow [0,1]$ and

ii A false membership function $f_A: X \rightarrow [0,1]$.

Where $t_A(x)$ is a lower bound of the grade of membership of x derived from the "evidence for x", and $f_A(x)$ is a lower bound on the negation of x derived from the "evidence against x" and $t_A(x) + f_A(x) \leq 1$. Thus the grade of membership of x in the vague set A is bounded by subinterval $[t_A(x), 1 - f_A(x)]$ of [0, 1]. The vague set A is written as $A = \{\langle x, [t_A(x), f_A(x)]\rangle / x \in \mathbb{W}\}$.

Where the interval $[t_A(x), 1 - f_A(x)]$ is called the value of x in the vague set A and denoted by $V_A(x)$.

Definition 2.9[5] A vague set A of a universe X with $t_A(x) = 0$ and $f_A(x) = 1$ for all $x \in \mathbb{W}$, is called the zero vague set of $\mathbb{W}$.

Definition 2.10[5] A vague set A of a universe X with $t_A(x) = 1$ and $f_A(x) = 0$ for all $x \in \mathbb{W}$ is called the zero vague set of $\mathbb{W}$.

Definition 2.11[5] Let A be a vague set of a universe X with the truth membership function t_A and the false membership functionf_A. For any $\alpha, \beta \in [0,1]$with $\alpha \leq \beta$, the $(\alpha, \beta) -$ cut of a vague set A is a crisp subset $A_{(\alpha,\beta)}$of the set X given by
$A_{(\alpha,\beta)} = \{x \in \mathcal{W} / V_A(x) \geq [\alpha, \beta]\}$.

Definition 2.12[5] The α-cut, A_α of the vague set is the (α, α)-cut of A and hence given by $A_\alpha = \{x \in \mathbb{W} / t_A(x) \geq \alpha\}$.

Definition 2.13[4] Let $I = [0,1]$ denote the family of all closed subintervals of $[0,1]$. If $I_1 = [a_1, b_1], I_2 = [a_2, b_2]$ are two elements of $I[0,1]$, we call $I_1 \geq I_2$if $a_1 \geq a_2$ and $b_1 \geq b_2$. We define the term *rmax* to mean the maximum of two intervals as
$rmax[I_1, I_2] = [\max\{a_1, a_2\}, \max\{b_1, b_2\}]$.

Similarly, we can define the term *rmin* of any two intervals.

Definition 2.14[5] The intersection of two vague sets A and B with respective truth membership functions and the false membership functions $t_A, t_B, f_A\ and\ f_B$ is a vague set $C = A \cap B$, whose truth membership function and false membership functions are related to those of A and B by
$t_C = \min\{t_A, t_B\}, 1 - f_C = \min\{1 - f_{A,} 1 - f_B\} = 1 - \max\{f_{A,}, f_{B,}\}$.

Definition 2.15[9] Let A be a vague set of lattice Wajsberg algebra $\mathbb{W}$. Then A is called a vague WI-ideal of $\mathbb{W}$, if the following axioms are satisfied for all $x, y \in \mathbb{W}$,

i $\quad V_A(0) \geq V_A(x)$,

ii $\quad V_A(x) \geq rmin\{V_A(x \to y)^*, V_A(y)\}$.

3.1. Vague positive implicative *W*-implicative ideal

In this section, we introduce the definition of vague positive implicative W-implicative ideal of lattice Wajsberg algebra, and investigate some related properties.

Definition 3.1.1 A vague set A of lattice Wajsberg algebra $\mathbb{W}$ is called a vague positive implicative W-implicative ideal of $\mathbb{W}$, if for all $x, y, z \in \mathbb{W}$,

i $\quad V_A(0) \geq V_A(x)$

ii $\quad V_A(y) \geq rmin\{V_A(((y \to (z \to y)^*)^* \to x)^*), V_A(x)\}$.

Example 3.1.2 Consider a set $\mathbb{W} = \{0, a, b, c, 1\}$ with partial ordering as in Figure 3.1.1. Defining a binary operation $' \to '$ and a quasi complement '$*$' on $\mathbb{W}$ as given in tables 3.1.1 and 3.1.2.

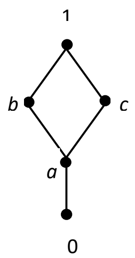

Figure: 3.1.1
Lattice diagram

x	x^*
0	1
a	b
b	a
c	c
1	0

Table: 3.1.1
Complement

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	a	a	c	1	1
b	c	b	1	1	1
c	b	1	1	1	1
1	0	a	b	c	1

Table: 3.1.2
Implication

Define " $\vee$ " and " $\wedge$ " operations on $\mathbb{W}$ as follows:
$(x \vee y) = (x \to y) \to y$
$(x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in \mathbb{W}$.

Then, $(\mathbb{W}, \vee, \wedge, *, 0, 1)$ is a lattice Wajsberg algebra.

Let A be a vague set of $\mathbb{W}$ defined by

$$A = \{\langle 0, [0.6, 0.3]\rangle, \langle a, [0.5, 0.2]\rangle, \langle b, [0.6, 0.2]\rangle, \langle c, [0.7, 0.2]\rangle, \langle 1, [0.7, 0.3]\rangle\}$$

Then, A is is vague positive implicative W-implicative ideal of $\mathbb{W}$.

Theorem 3.1.3 Every vague positive implicative W-implicative ideal of lattice Wajsberg algebra $\mathbb{W}$ is a vague W-implicative ideal of $\mathbb{W}$.

Proof: Let A be a vague positive implicative W-implicative ideal of $\mathbb{W}$.

Then from (ii) of definition 3.1.1,

we have $V_A(y) \geq rmin\{V_A(((y \to (z \to y)^*)^* \to x)^*), V_A(x)\}$ for all $x, y, z \in \mathbb{W}$. (3.1.1)

Taking $x = y, y = x,$ and $z = x$ in (3.1.1), we get

$$V_A(x) \geq rmin\{V_A(((x \to (x \to x)^*)^* \to y)^*), V_A(y)\}$$

$$= rmin\{V_A(((x \to 1)^* \to)^*)^*, V_A(y)\}$$

$$= rmin\{V_A(((x \to 0)^* \to)^*)^*, V_A(y)\}$$

$$= rmin\{V_A((x \to y))^*, V_A(y)\}$$

Thus, $V_A(x) \geq rmin\{V_A((x \to y))^*, V_A(y)\}$, and $V_A(0) \geq V_A(x)$. ■

Note. The converse of the above proposition may not be true.

Proposition 3.1.4 Let V_A be a vague implicative W-implicative ideal of lattice Wajsberg algebra $\mathbb{W}$. V_A is a vague positive implicative W-implicative ideal of $\mathbb{W}$ if and only if $V_A(x) \geq V_A(((x \to (y \to x)^*)^*$ for all $x, y \in \mathbb{W}$.

Proof: Let V_A be a vague positive implicative W-implicative ideal of $\mathbb{W}$, then from (ii) of definition 3.1.1 we have

$V_A(y) \geq rmin\{V_A(((y \to (z \to y)^*)^* \to x)^*), V_A(x)\}$ for all $x, y, z \in \mathbb{W}$. (3.1.2)

Substituting $x = 0, y = x$ and $z = y$ in (3.1.2) we get

$$V_A(x) \geq rmin\{V_A(((x \to (y \to x)^*)^* \to 0)^*)^*, V_A(0)\}$$

$$= rmin\{V_A((x \to (y \to x)^*), V_A(0)\}$$

$$= V_A((x \to (y \to x)^*)^*)$$

Conversely, suppose V_A is a vague W-implicative ideal and it satisfies the inequality,

$V_A(x) \geq V_A((x \to (y \to x)^*)^*)$ for all $x, y, z \in \mathbb{W}$ (3.1.3)

Put $x = y$ in (3.1.3) then, we have

$$V_A(y) \geq V_A((y \to (z \to y)^*)^*)$$

$$\geq rmin\{V_A(((y \to (z \to y)^*)^* \to x)^*), V_A(x)\}$$

Thus, we have $V_A(y) \geq rmin\{V_A(((y \to (z \to y)^*)^* \to x)^*), V_A(x)\}$, and

$V_A(0) \geq V_A(x)$ [From (i) of definition 3.1.1]

Hence, V_A is a vague positive implicative W-implicative ideal of $\mathbb{W}$. ■

Proposition 3.1.5 If V_A is a vague positive implicative W-implicative ideal of lattice Wajsberg algebra $\mathbb{W}$ then, $I = \{x \in A / V_A(x) = V_A(0)\}$ is a positive implicative W-implicative ideal of $\mathbb{W}$.

Proof: Let V_A be a vague positive implicative W-implicative ideal of $\mathbb{W}$ and $I = \{x \in A/V_A(x) = V_A(0)\}$.

Obviously, $0 \in A$.

Let $((y \to (z \to y)^*)^* \to x)^*) \in I, x \in I$ for all $x, y, z \in \mathbb{W}$

Then, we have $V_A(((y \to (z \to y)^*)^* \to x)^*) = V_A(0)$ and $V_A(x) = V_A(0)$ (3.1.4)

Since V_A is a vague positive implicative W-implicative ideal, we have

$V_A(y) \geq \ rmin\{V_A(((y \to (z \to y)^*)^* \to x)^*), V_A(x)\}$ [From (ii) of definition 3.1.1]

$= \ V_A(0)$ [From 3.1.4]

and $V_A(0) \geq V_A(y)$ [From (i) of definition 3.1.1]

Then, we get $V_A(y) = V_A(0)$

Thus, $y \in I$ it follows that I is a positive implicative W-implicative ideal of $\mathbb{W}$. ■

Theorem 3.1.6 Let V_A be a vague subset of lattice Wajsberg algebra $\mathbb{W}$. V_A is a vague positive implicative W-implicative ideal of $\mathbb{W}$ if and only if $V_A(\alpha, \beta) \neq \emptyset;\ \alpha, \beta \in [0,1]$.

Proof: Let V_A is a vague positive implicative W-implicative ideal of $\mathbb{W}$ and $\alpha, \beta \in [0,1]$ such that $V_A(\alpha, \beta) \neq \emptyset$. Clearly $0 \in V_A(\alpha, \beta)$.

Let $((y \to (z \to y)^*)^* \to x)^*) \in \ V_A(\alpha, \beta)$ and $x \in \ V_A(\alpha, \beta)$ for all $x, y, z \in \mathbb{W}$

Then, we have $V_A(((y \to (z \to y)^*)^* \to x)^* \geq [\alpha, \beta], V_A(x) \geq [\alpha, \beta]$.

It follows that, $V_A(y) \geq rmin\{V_A(((y \to (z \to y)^*)^* \to x)^*), V_A(x)\} \geq [\alpha, \beta]$.

Thus, $y \in V_A[\alpha, \beta]$.

Hence, $[\alpha, \beta]$ is a positive implicative W-implicative ideal of $\mathbb{W}$.

Conversely, if $V_A(\alpha, \beta) \neq \emptyset$ is a positive implicative W-implicative ideal of $\mathbb{W}$, where $\alpha, \beta \in [0,1]$. For any $x \in \mathbb{W}$ and $x \in V_A(A)$, it follows that $V_A(A)(x)$ is a positive implicative W-implicative ideal of $\mathbb{W}$.

Thus, $0 \in V_A(A)(x)$. That is, $V_A(0) \geq V_A(x)$ for all $x, y, z \in \mathbb{W}$.

Let $[\alpha, \beta] = rmin\{V_A(((y \to (z \to y)^*)^* \to x)^*)^*, V_A(x)\}$, it follows that $V_A(\alpha, \beta)$ is a positive implicative W-implicative ideal and $((y \to (z \to y)^*)^* \to x)^*)^* \in V_A[\alpha, \beta]$ and $x \in V_A[\alpha, \beta]$.

This implies that $y \in V_A[\alpha, \beta]$.

So, $V_A(y) \geq [\alpha, \beta] = rmin\{V_A(((y \to (z \to y)^*)^* \to x)^*, V_A(x)\}$

Thus, V_A is a vague positive implicative W-implicative ideal of $\mathbb{W}$. ■

Corollary 3.1.7 A vague subset V_A of lattice Wajsberg algebra $\mathbb{W}$ is a vague positive implicative W-implicative ideal of $\mathbb{W}$ if and only if V_α is a positive implicative W-implicative ideal of $\mathbb{W}$, when $V_\alpha \neq \emptyset, \alpha \in [0,1]$.

Proposition 3.1.8 Let M and N be implicative W-implicative ideals of lattice Wajsberg algebra $\mathbb{W}$, such that $M \subseteq N$. If V_A is a vague positive implicative W-implicative ideal of M. Then so on N.

Proof: Let M and N be implicative W-implicative ideals of lattice Wajsberg algebra $\mathbb{W}$. Let V_A be a vague positive implicative W-implicative ideal of M.

Since $M \subseteq N$, $V_M(x) \leq V_N(x)$ for all $x \in \mathbb{W}$.

Then, clearly $M_\alpha \leq N_\alpha$ for every $\alpha \in [0,1]$.

If V_M is a vague positive implicative W-implicative ideal of $\mathbb{W}$.

Hence, we get M_α is a positive implicative W-implicative ideal of $\mathbb{W}$.

[From corollary 3.1.7]

Then, N_α is a positive implicative W-implicative ideal of $\mathbb{W}$. [From Proposition 2.10]

Thus, V_N is a positive implicative W-implicative ideal.

Hence V_A is a vague positive implicative W-implicative ideal of N. ■

3.2. Vague Associative *W*-implicative ideal

In this section,we introduce an notation of vague associative W-implicative ideal of lattice Wajsberg algebra $\mathbb{W}$ and examine its properties.

Definition 3.2.1 A vague subset V_A of lattice Wajsberg algebra $\mathbb{W}$ is said to be a vague associative W-implicative ideal of $\mathbb{W}$ with respect to x, where x is a fixed element of $\mathbb{W}$, if it satisfies,

i $V_A(0) \geq V_A(y)$

ii $V_A(z) \geq rmin\{V_A((z \to y)^* \to x)^*, V_A((y \to x)^*)\}$ for all $x, y, z \in \mathbb{W}$

Note. A vague associative W-implicative ideal with respect to all $x \neq 1$ is called a vague associative W-implicative ideal. Vague associative W-implicative ideal with respect to 1 is constant.

Example 3.2.2 Consider a set $A = \{0, f, g, 1\}$ with partial ordering as in Figure 3.2.1. Define '*' and '→' on $\mathbb{W}$ as given in tables 3.2.1 and 3.2.2.

→	0	f	g	1
0	1	1	1	1

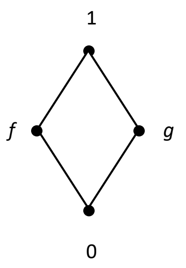

Figure 3.2.1
Lattice diagram

x	x^*
0	1
f	g
g	f
1	0

Table 3.2.1
Complement

f	f	1	1	1
g	f	g	1	1
1	0	f	g	1

Table 3.2.2
Implication

Here, $\mathbb{W}$ is a lattice Wajsberg algebra.

A vague subset V_A of $\mathbb{W}$ is defined by,

$$A = \{\langle 0, [0.7,0.2]\rangle, \langle f, [0.7,0.2]\rangle, \langle g, [0.5,0.3]\rangle, \langle 1, [0.5,0.3]\rangle\}.$$

Then, V_A is a vague associative W-implicative ideal of $\mathbb{W}$.

Proposition 3.2.3 If V_A is a vague associative W-implicative ideal of $\mathbb{W}$ with respect to x then $V_A(0) = V_A(y)$.

Proof: Let V_A be a vague associative W-implicative ideal of $\mathbb{W}$ with respect to x if $x = (0, 1)$. Then it is trivial.

If x is neither 0 nor 1. Then,

$$V_A(x) \geq rmin\{V_A((x \to 0)^* \to x)^*, V_A((y \to x)^*)\}$$ [From (ii) of definition 3.2.1]

Thus, $V_A(x) = V_A(0)$. ■

Proposition 3.2.4 Every vague associative W-implicative ideal of lattice Wajsberg algebra $\mathbb{W}$ with respect to 0 is a vague W-implicative ideal of $\mathbb{W}$.

Proof: Let V_A be a vague associative W-implicative ideal of $\mathbb{W}$ with respect to 0.

Then, we have $V_A(x) \geq rmin\{V_A((x \to y)^* \to 0)^*, V_A((y \to 0)^*)\}$ for all $x, y \in \mathbb{W}$

[From (ii) of definition 3.2.1]

$$= rmin\{V_A((x \to y)^*), V_A(y)\}$$

Thus, V_A is a vague W-implicative ideal of $\mathbb{W}$. ■

Theorem 3.2.5 Let V_A be a vague W-implicative ideal of lattice Wajsberg algebra of $\mathbb{W}$. V_A is a vague associative W-implicative ideal of $\mathbb{W}$ if and only if it satisfies, $V_A((z \to (y \to x)^*)^* \geq V_A((z \to y)^* \to x)^*$ for all $x, y, z \in \mathbb{W}$.

Proof: Let V_A be a vague W-implicative ideal of $\mathbb{W}$ satisfying

$$V_A((z \to (y \to x)^*)^* \geq V_A((z \to y)^* \to x)^* \text{ for all } x, y, z \in \mathbb{W}$$

Then, $V_A(z) \geq rmin\{V_A((z \to (y \to x)^*)^*), V_A((y \to x)^*)\}$

$= rmin\{V_A(((z \to y)^* \to x)^*), V_A((y \to x)^*)\}$

Thus, V_A is a vague associative W-implicative ideal of $\mathbb{W}$.

Conversely, if V_A be a vague associative W-implicative ideal of $\mathbb{W}$.

Then, $V_A((z \to (y \to x)^*)^*) \geq rmin\{V_A(((z \to (y \to x)^*)^* \to (z \to y)^*)^* \to x), V_A((z \to (y \to x)^*)^*)\}$

Let us consider, $(((z \to (y \to x)^*)^* \to (z \to y)^*)^* \to x)$

$= x^* \to ((z \to (y \to x)^*)^* \to (z \to y)^*)$

$= x^* \to ((z \to y) \to (z \to (y \to x)^*))$

$= (x \to y) \to (x^* \to ((y \to x) \to z^*)$

$= (z \to y) \to ((y \to x) \to (x^* \to z^*))$

$= (z \to y) \to ((z \to y) \to (z \to y))$

$= 1$

It follows that $V_A((z \to (y \to x)^*)^*) \geq rmin\{V_A(0), V_A(((z \to y)^* \to x)^*)\}$

$= V_A(((z \to y)^* \to x^*)$

Thus, $V_A((z \to (y \to x)^*)^*) \geq V_A(((z \to y)^* \to x^*)$. ■

Theorem 3.2.6 Let V_A be a vague W-implicative ideal of lattice Wajsberg algebra $\mathbb{W}$. V_A is a vague associative W-implicative ideal of $\mathbb{W}$ if and only if $V_A(z) \geq V_A(((z \to x)^* \to x^*)$ for all $x, y, z \in \mathbb{W}$.

Proof: Let V_A be a vague associative W-implicative ideal of $\mathbb{W}$.

Then, $V_A(z) \geq rmin\{V_A((z \to y)^* \to x)^*, V_A((y \to x)^*)\}$ for all $x, y, z \in \mathbb{W}$.
[From (ii) of definition 3.2.1]

Taking $y = x$ we get,

$$V_A(z) \geq rmin\{V_A((z \to x)^* \to x)^*, V_A((x \to x)^*)\}$$

$$= rmin\{V_A((z \to y)^* \to x)^*, V_A(0)\}$$

$$= V_A(((z \to y)^* \to x^*)$$

Conversely, if V_A is a vague W-implicative ideal and satisfies

$V_A(z) \geq V_A(((z \to y)^* \to x)^*)$ for all $x, y, z \in \mathbb{W}$

Clearly, $(((z \to x)^*) \to (y \to x)^*)^* \to (z \to y)^*)^* = 0$

and $((z \to y)^* \to (z \to x)^*)^* \le (x \to y)^*$

It follows that, $(((z \to (y \to x)^*)^* \to x)^* \to ((z \to y)^* \to x)^*)^* = 0$

$V_A((z \to (y \to x)^*)^* \ge V_A(((z \to y \to x)^*) \to x)^* \to x)^*)$

$\ge rmin\{V_A(((z \to (y \to x)^* \to x)^*, V_A((z \to y)^* \to x)^*)^*\}$

$V_A(((z \to y)^* \to x)^*) = rmin\{V_A(0), V_A(((z \to y)^* \to x)^*)$

$= V_A(((z \to y)^* \to x)^*)$

From the proposition 3.2.3, V_A is a vague associative W-implicative ideal of $\mathbb{W}$. ■

4. Conclusions

In this paper, we have introduced the notions of vague positive implicative W-implicative ideal and vague associative W-implicative ideal of lattice Wajsberg algebras. Further, we have discussed the relationship between vague positive implicative W-implicative ideal, and vague W-implicative ideal. Moreover, we have given some of the characterization of the vague associative W-implicative ideal.

References

[1] Anitha, T., and Amarendra Babu, V., Vague *positive implicative and vague associative LI-ideals of lattice implication algebras*, International Journal of Pure and Applied Mathematics Volume 105, No. 1 (2015), 39-57.

[2] Anitha, T., and Amarendra Babu, V., *Vague LI - ideals on lattice implication algebras*, J. Emerging Trends in Computing and Information Sciences, 5, (2014), 788-793.

[3] Anitha, T., and Amarendra Babu,V., *Vague implicative LI - ideals of lattice implication algebras*, Mathematics and statistics, 3, No. 3, (2015), 53-57.

[4] Font, J. M., Rodriguez, A. J., and Torrens, A., *Wajsberg algebras*, STOCHASTICA Volume 8, Number 1, (1984), 5-31.

[5] Gau, W. L., Buehrer, D. J., *Vague sets*, IEEE Transactions on Systems, Man and Cybernetics, 23(20)(1993), 610-614.

[6] Ibrahim, A., and Shajitha Begum, C., *On WI-ideals of lattice Wajsberg algebras*, Global Journal of Pure and Applied Mathematics, Volume 13, Number 10 (2017), 7237-7254.

[6] Ibrahim, A., and Shajitha Begum, C., *Ideals and implicative WI-ideals of Lattice Wajsberg algebras*, IPASJ International Journal of Computer Science, Volume 6, Issue 4 (2018), 30-38.

[7] Ibrahim, A., and Shajitha Begum, C., *Fuzzy implicative WI-ideals of Lattice Wajsberg algebras*, Journal of Computer and Mathematical Science, Volume 9, Number 8 (2018), 1026 -1035.

[8] Ibrahim, A., and Mohamed Rajik, M., *VWI-ideals of lattice Wajsberg algebras,* Journal of Xidian University, Volume 14, Number 3(2020), 1284-1289.

[9] Wajsberg, M., *Beitragezum Metaaussagenkalkull*,Monat. Mat. phys. 42 (1935), 221-242.

[10] Zadeh, L. A., *Fuzzy sets*, Information and Control 8 (1965), 338-353.