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# On geometrical hyperstructures of finite order

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**Abstract**

It is known that a concrete representation of a finite *k*-dimensional Projective Geometry can be given by means of marks of a Galois Field GF $[p^n]$, denoted by PG(k, $p^n$).
In this geometry, we define hyperoperations, which create hyperstructures of finite order and we present results, propositions and examples on this topic. Additionally, we connect these hyperstructures to Join Spaces.

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## 1. Introduction

The algebraic hyperstructures, which constitute a generalization of the ordinary algebraic structures, were introduced by Marty in 1934 [5]. Since then, many researchers worked on hyperstructures. The results of this work, as well as, applications of the hyperstructures theory can be found in the books [2] and [3]. Vougiouklis in 1991 introduced a larger class than the known hyperstructures, so called $H_v$-structures [8] and all about them can be found in his book [9].
Let us give some basic definitions, appearing in [3], [9]:

Let $H$ be a set, $P'(H)$ the family of nonempty subsets of $H$ and $(\cdot)$ a hyperoperation in $H$, that is

$$\cdot : H\times H \to P'(H)$$

If $(x,y)\in H\times H$, its image under $(\cdot)$ is denoted by $x\cdot y$ or $xy$. If $A$ , $B \subseteq H$ then $A\cdot B$ or $AB$ is given by $AB = \cup \{xy / x\in A , y\in B\}$.
$xA$ is used for $\{x\}A$ and $Ax$ for $A\{x\}$. Generally, the singleton $\{x\}$ is identified with its member x.
The hyperoperation $(\cdot)$ is called *associative* in $H$ if

$$(xy)z = x(yz) \text{ for all } x,y,z\in H$$

The hyperoperation $(\cdot)$ is called *commutative* in $H$ if

$$xy = yx \text{ for all } x,y\in H$$

A hypergroupoid $(H,\cdot)$ that satisfies reproducibility, $xH = Hx = H$ for all $x\in H$, and associativity, is called *hypergroup*.
A *join operation* $(\cdot)$ [6] in a set J is a mapping of $J\times J$ into the family of subsets of J. A *join space* is defined as a system $(J,\cdot)$, where $(\cdot)$ is a join operation in the arbitrary set J, which satisfies the postulates:

$$\text{i) } a\cdot b\neq\emptyset \quad \text{ii) } a\cdot b = b\cdot a \quad \text{iii) } (a\cdot b)\cdot c=a\cdot(b\cdot c)$$

$$\text{iv) } a/b \cap c/d\neq\emptyset \Rightarrow a\cdot d \cap b\cdot c\neq\emptyset \quad \text{v) } a/b = \{x\in J / a\in b\cdot x\}\neq\emptyset .$$

The $H_v$-structures are hyperstructures satisfying the weak axioms, where the non-empty intersection replaces the equality.

Let $H\neq\emptyset$ be a set equipped with the hyperoperations $(+)$, $(\cdot)$, then the *weak associativity* in $(\cdot)$ is given by the relation

$$(x\cdot y)\cdot z \cap x\cdot(y\cdot z) \neq \emptyset, \ \forall x,y,z\in H.$$

The $(\cdot)$, is called *weak commutative* if

$$x\cdot y \cap y\cdot x \neq \emptyset, \ \forall x,y\in H.$$

The hyperstructure $(H,\cdot)$ is called *$H_v$-semigroup* if $(\cdot)$ is weak associative and it is called *$H_v$-quasigroup* if the reproduction axiom is valid, i.e. $x\cdot H=H\cdot x=H, \ \forall x\in H$.

The hyperstructure $(H,\cdot)$ is called *$H_v$-group* if it is an $H_v$-quasigroup and an $H_v$-semigroup. It is called *$H_v$-commutative group* if it is an $H_v$-group and the weak commutativity is valid.

The *weak distributivity* of $(\cdot)$ with respect to $(+)$ is given for all $x,y,z \in H$, by

$$x\cdot(y+z) \cap (x\cdot y+x\cdot z) \neq \varnothing \ , \ (x+y)\cdot z \cap (x\cdot z+y\cdot z) \neq \varnothing.$$

Using these axioms, the *$H_v$-ring*, which is the largest class of algebraic systems that satisfy ring-like axioms, is defined to be the triple $(H,+,\cdot)$, where in both $(+)$ and $(\cdot)$ the weak associativity is valid, the weak distributivity is also valid and $(+)$ is *reproductive*, i.e $x+H = H+x = H$, $\forall x \in H$.

An $H_v$-ring $(R,+,\cdot)$ is called *dual $H_v$-ring* if the hyperstructure $(R,\cdot,+)$ is also an $H_v$-ring [4].

Let $(H,\cdot)$ be a hypergroup or an $H_v$-group. The $\beta^*$ relation is defined as the smallest equivalence relation, one can say also congruence, such that, the quotient $H/\beta^*$ is a group.

The $\beta^*$ is called *fundamental equivalence relation*.

## 2. Representation of the geometry of a *k*-dimensional space by means of Galois Fields

Veblen and Bussey [7] have defined a finite projective geometry, which is said to be a geometry of a $k$-dimensional space, in the following way.

It consists of a set of elements, called points for suggestiveness, which are subjected to the following five conditions or postulates:

I. The set contains a finite number of points. It contains one or more subsets called lines, each of which contains at least three points.

II. If A and B are distinct points, there is one and only one line that contains both A and B.

III. If A, B, C are noncollinear points and if a line $l$ contains a point D of the line AB and a point E of the line BC but does not contain A or B or C, then the line $l$ contains a point F of the line CA.

IV. If $m$ is an integer less than $k$, not all the points considered are in the same $m$-space.

V. If (IV) is satisfied, there exists in the set of points considered no ($k$+1)-space.

Furthermore, a point is called *0*-space, a line is called *1*-space and a plane is called *2*-space.

By means of marks of a Galois field, we shall now give a concrete representation of a finite $k$-dimensional projective geometry.

We denote a point of the geometry by the ordered set of coordinates $(\mu_0, \mu_1, \mu_2, \ldots, \mu_k)$, where $\mu_0, \mu_1, \mu_2, \ldots, \mu_k$ are marks of the GF[$p^n$], at least one of which is different from zero. It is understood that the foregoing symbol $(\mu_0, \mu_1, \mu_2, \ldots, \mu_k)$ denotes the same point as the symbol $(\mu\mu_0, \mu\mu_1, \mu\mu_2, \ldots, \mu\mu_k)$, where $\mu$ is one of the $p^n$-1 nonzero marks of the field.

The ordered set of marks $\mu_0, \mu_1, \mu_2, \ldots, \mu_k$ may be chosen in $(p^n)^{\kappa+1}$ ways, but since the symbol (0, 0, 0,….,0) is excepted, then it may be chosen in $(p^n)^{\kappa+1}-1$ ways. So, there exists $(p^n)^{\kappa+1}-1$ points. In this totality, each point is represented in $p^n-1$ ways (there are $p^n-1$ nonzero marks in the field) and thus, it follows that the number of points defined is

$$\frac{(p^n)^{k+1}-1}{p^n-1} = 1 + p^n + \ldots.. + p^{kn}$$

This representation of the finite $\kappa$-dimensional projective geometry by means of the marks of the GF[$p^n$] constitute the projective geometry PG($\kappa$, $p^n$) [1].

Now, for the line containing the two distinct points $(\mu_0, \mu_1, \mu_2, \ldots, \mu_k)$ and $(\nu_0, \nu_1, \nu_2, \ldots, \nu_k)$ we consider the set of points :

$$(\mu\mu_0 + \nu\nu_0 , \mu\mu_1 + \nu\nu_1 , \mu\mu_2 + \nu\nu_2 , \ldots\ldots, \mu\mu_k + \nu\nu_k )$$

where $\mu$ and $\nu$ run independently over the marks of the GF[$p^n$], subjected to the condition that $\mu$ and $\nu$ shall not be simultaneously zero.

Then the number of possible combinations of $\mu$ and $\nu$ is $(p^n)^2 - 1$ and for each of these the corresponding symbol denotes a point, since not all the k+1 coordinates are zero. But the same point is represented $p^n$-1 times, due to the factor of proportionality involved in the definition of points. Therefore, a line so defined contains

$$\frac{(p^n)^2-1}{p^n-1}=p^n+1 \text{ points.}$$

It is obvious that any two points on the line may be used in this way to define the same line.
The five postulates given above for the *k*-dimensional space are satisfied by the concrete elements thus introduced [1].

## 3. On a hypergroup of finite order

Let us denote by V the set of the elements of the PG(κ, $p^n$) and for x,y∈V let us denote by $l_{xy}$ the line which is defined by the points x and y. By $l_{x-}$ is denoted the line which is defined by the point x and any other point of V.
We define the hyperoperation (·) on V, as follows :

**Definition 1.** For every x,y∈V , $\cdot : V\times V \to P'(V)$ , such that

$$x\cdot y = \begin{cases} x & if \;\; x=y \\ l_{xy} & if \;\; x\neq y \end{cases}$$

Obviously, the above hyperoperation is a commutative one, since

$$x\cdot y = l_{xy} = l_{yx} = y\cdot x \quad \text{for every } x,y\in V \text{ and } x \neq y.$$

One can compare the above defined hyperoperation with the join operation [6], when Euclidean Geometry is converted into Join Spaces by defining *ab* with $a\neq b$, to be the open segment, whose endpoints are *a* and b. Moreover, *aa* is defined to be *a*.

**Proposition 2.** For every noncollinear x,y,z∈V, $|x\cdot(y\cdot z)|=|(x\cdot y)\cdot z| = p^{2n}+p^n+1$.

**Proof.** All the lines defined in V are having one point in common, at most. First, let us calculate the points of the set $x\cdot(y\cdot z)$. For $y \neq z$, the set $y\cdot z = l_{yz}$ consists of $p^n+1$ points, including $y$ and $z$. On the other hand, the point $x$ $(x \neq y,z)$, with each of the $p^n+1$ points of the line $l_{yz}$, creates $p^n+1$ lines of the type $l_{x-}$ which are having the point $x$ in common. This means that the $p^n+1$ lines of the type $l_{x-}$ are having no other point in common. So, each line $l_{x-}$ is having $p^n$ different points from the others. Then it follows that the set $x\cdot(y\cdot z)$ consists of $(p^n+1)\cdot p^n+1 = p^{2n}+p^n+1$ different points.

Similarly, it arises that $|(x\cdot y)\cdot z| = p^{2n}+p^n+1$.■

**Proposition 3.** The hyperstructure $(V, \cdot)$ is a hypergroup.
**Proof.** Easily follows, that for every $x \in V$

$$x\cdot V = \bigcup_{v\in V}(x\cdot v) = \bigcup_{v\in V}(v\cdot x) = V\cdot x = V$$

Now, for every $x, y, z \in V$

if $x = y = z$ then $x\cdot(y\cdot z) = (x\cdot y)\cdot z = x$
if $x = y \neq z$ then $x\cdot(y\cdot z) = (x\cdot y)\cdot z = l_{xz}$
if $x = z \neq y$ then $x\cdot(y\cdot z) = (x\cdot y)\cdot z = l_{xy}$
if $y = z \neq x$ then $x\cdot(y\cdot z) = (x\cdot y)\cdot z = l_{xy}$
if $x \neq y \neq z$ and $x, y, z$ collinear, then $x\cdot(y\cdot z) = (x\cdot y)\cdot z = l_{xy}$
if $x, y, z$ noncollinear then

for the line $l_{yz}$ containing the two distinct points $y(y_0, y_1,..,y_k)$ and $z(z_0, z_1,\ldots, z_k)$ we take the set of points :

$$(\mu y_0 + \nu z_0 , \mu y_1 + \nu z_1 ,\ldots\ldots, \mu y_k + \nu z_k )$$

where $\mu$ and $\nu$ run independently over the marks of the $GF[p^n]$ subjected to the condition that $\mu$ and $\nu$ shall not be simultaneously zero.
Let $x(x_0, x_1,...,x_k)$. Then for the set $x\cdot(y\cdot z)$ we take the set of points :

$$\big(\rho x_0 + \lambda(\mu y_0 + \nu z_0) , \rho x_1 + \lambda(\mu y_1 + \nu z_1) ,\ldots\ldots, \rho x_k + \lambda(\mu y_k + \nu z_k)\big) \quad (1)$$

where $\rho$ and $\lambda$ run independently over the marks of the $GF[p^n]$ subjected to the condition that $\rho$ and $\lambda$ shall not be simultaneously zero.
Let $w \in x\cdot(y\cdot z)$ , then the coordinates of the point $w$ is of the form of (1).
For some $i \in 0,1,\ldots,k$ we have

$$\rho x_i+\lambda(\mu y_i+\nu z_i) = \rho x_i+\lambda\mu y_i+\lambda\nu z_i = \rho x_i+\lambda\mu y_i+\nu' z_i, \text{ where } \nu' \in GF[p^n]$$

If $\rho=0$ (2) then $\rho=\lambda 0$ for every $\lambda \in GF[p^n]$ and then

$$\rho x_i + \lambda\mu y_i + \nu' z_i = \lambda 0 x_i + \lambda\mu y_i + \nu' z_i = \lambda(0x_i + \mu y_i) + \nu' z_i$$

If $\rho \neq 0$ (3) then $\rho=\lambda\mu'$ for every $\lambda, \mu' \in GF[p^n]$-{0} and then

$$\rho x_i + \lambda\mu y_i + \nu' z_i = \lambda\mu' x_i + \lambda\mu y_i + \nu' z_i = \lambda(\mu' x_i + \mu y_i) + \nu' z_i$$

The coordinates of the points of the set $(x\cdot y)\cdot z$ are of the form

$$\kappa(\tau x_i + \tau' y_i) + \kappa' z_i$$

where $\tau$, $\tau'$, $\kappa$, $\kappa'$ run independently over the marks of the $GF[p^n]$ subjected to the condition that $\tau$, $\tau'$ and $\kappa$, $\kappa'$ shall not be simultaneously zero.
Due to the conditions (2) and (3) we get that

$w \in x\cdot(y\cdot z) \Rightarrow w \in (x\cdot y)\cdot z$ which means that $x\cdot(y\cdot z) \subset (x\cdot y)\cdot z$ .

In a similar way, it can be proven that for $w' \in (x\cdot y)\cdot z \Rightarrow w' \in x\cdot(y\cdot z)$, which means that $(x\cdot y)\cdot z \subset x\cdot(y\cdot z)$. So,

$$x\cdot(y\cdot z) = (x\cdot y)\cdot z \quad \text{for every } x,y,z \in V. \blacksquare$$

**Remark 4.** For the hypergroup $(V,\cdot)$, since $\{x,y\} \subset x\cdot y \ \ \forall x,y \in V$, the $V/\beta^*$ is a singleton.

**Proposition 5.** The hypergroup $(V, \cdot)$ is a Join Space.
**Proof.** Since the hyperoperation $(\cdot)$ is commutative, the hyperstructure $(V, \cdot)$ is a commutative hypergroup.

Moreover, let $a/b \cap c/d \neq \varnothing$ , $a,b,c,d \in V$ . Then, there exists $w \in V$ such that $w \in a/b$ which implies that $a \in l_{bw}$ and $w \in c/d$ which implies that $c \in l_{dw}$. Since the lines of V are having one point in common at most, the lines $l_{bw}$ and $l_{dw}$ intersect at w.
Let the ordered set of coordinates of the points w, a, d be $(w_0, w_1,...,w_k)$ , $(a_0, a_1,...,a_k)$, $(d_0, d_1,...,d_k)$ respectively. Then, the coordinates of the point b will be of the form $(\lambda a_0 + \mu w_0 , \lambda a_1 + \mu w_1 ,\ldots\ldots, \lambda a_k + \mu w_k )$, where $\lambda,\mu \in GF[p^n]$ and the coordinates of the point c will be of the form $(\kappa d_0 + \rho w_0 , \kappa d_1 + \rho w_1,\ldots \ldots\ldots, \kappa a_k + \rho w_k )$, where $\kappa,\rho \in GF[p^n]$. Since the points w, a, d do not belong to the line $l_{bc}$ , the marks $\lambda,\mu,\kappa,\rho$ of the $GF[p^n]$ are not zero.
Now, $l_{bc}$ consists of the points of the form

$$(\nu\lambda a_0+\nu\mu w_0+ \tau\kappa d_0+\tau\rho w_0 , \nu\lambda a_1+\nu\mu w_1+ \tau\kappa d_1+\tau\rho w_1 ,\ldots\ldots, \nu\lambda a_k+\nu\mu w_k+ \tau\kappa d_k+\tau\rho w_k ),$$

where $\nu$ and $\tau$ run independently over the marks of the $GF[p^n]$ and they are not simultaneously zero.
It is known that for the nonzero marks $\mu$ and $\rho$, there exist nonzero marks $\nu$ and $\tau$ such that: $\nu\mu+\tau\rho = 0$ . Then, we get
$\nu\mu w_0 + \tau\rho w_0 = (\nu\mu+\tau\rho)w_0 = 0$, $\nu\mu w_1 + \tau\rho w_1 = (\nu\mu+\tau\rho)w_1 = 0$ ,…….., $\nu\mu w_k + \tau\rho w_k = (\nu\mu+\tau\rho)w_k = 0$ .
In that case, the point $(\nu\lambda a_0+\tau\kappa d_0 , \nu\lambda a_1+\tau\kappa d_1 ,\ldots\ldots., \nu\lambda a_k+\tau\kappa d_k)$ of the line $l_{bc}$ is additionally a point of the line $l_{ad}$. So, the lines $l_{bc}$, $l_{ad}$ intersect and then:

$$a\cdot d \cap b\cdot c \neq \varnothing \text{ for all } a,b,c,d \in V . \blacksquare$$

## 4. On a $H_v$-group of finite order

Now, we define a new hyperoperation $(\circ)$ on V as follows :

**Definition 6.** For every $x,y \in V$ , $\circ : V\times V \to P'(V)$ , such that

$$x\circ y = \begin{cases} x & if \;\; x = y \\ l_{xy} - \{x\} & if \;\; x \neq y \end{cases}$$

Every line of the set V contains $p^n+1$ points. The hyperoperation ($\circ$) is weak commutative, since the lines $l_{xy}-\{x\}$ and $l_{yx}-\{y\}$ are having $p^n+1-2 = p^n-1$ points in common, so

$$(x\circ y) \cap (y\circ x) \neq \varnothing \quad \text{for every } x,y\in V .$$

**Proposition 7.** For every noncolliner $x,y,z\in V$,

$$|x\circ(y\circ z)| = p^{2n} > |(x\circ y)\circ z| = p^{2n}-p^n+1$$

**Proof.** Since the line $y\circ z$ does not contain the point y, we get that $|y\circ z| = p^n$. The point x creates $p^n$ points with each of the $p^n$ points of the line $y\circ z$ (the point x is not participating according to the hyperoperation ($\circ$)). So, $|x\circ(y\circ z)| = p^n\cdot p^n = p^{2n}$.
On the other hand, the line $x\circ y$ consists of $p^n$ points. Each of these points creates $p^n$ points each time together with the point z, but since the point z appears $p^n$ times, we get

$$|(x\circ y)\circ z| = p^n\cdot p^n - p^n+1 = p^{2n}-p^n+1.$$

Since p is prime and $n\in IN$, easily follows that $|x\circ(y\circ z)| > |(x\circ y)\circ z|$.■

**Proposition 8.** The hyperstructure $(V, \circ)$ is an $H_v$-group.
**Proof.** Indeed, for every $x\in V$

$$x\circ V = \bigcup_{v\in V}(x\circ v) = (x\circ x) \cup \left(\bigcup_{v\in V-\{x\}}(x\circ v)\right) = x\cup\left(\bigcup_{v\in V-\{x\}}(l_{xv}-\{x\})\right) = V$$

since every line $l_{xv}$ always contains the point $v\in V$.
On the other hand, for every $x\in V$

$$V\circ x = \bigcup_{v\in V}(v\circ x) = (x\circ x) \cup \left(\bigcup_{v\in V-\{x\}}(v\circ x)\right) = x\cup\left(\bigcup_{v\in V-\{x\}}(l_{vx}-\{v\})\right) = V$$

Indeed, having the fact that every line of V contains at least 3 points, for every line $l_{vx}-\{v\}=v\circ x$ there exists at least one point $v'\in l_{vx}-\{v\}$, that $v\in l_{v'x}-\{v'\} = v'\circ x$. So,

$$x\circ V = V\circ x = V \quad \text{for every } x\in V.$$

The hyperoperation ($\circ$) is weak associative, since for every $x,y,z \in V$

$$x\circ(y\circ z) \supset x\circ z \ni z \quad \text{and} \quad (x\circ y)\circ z \supset y\circ z \ni z$$

But, we shall go further on, proving that the inclusion on the right parenthesis is valid, i.e $(x\circ y)\circ z \subset x\circ(y\circ z)$.

From the proposition 7 we get that $|x\circ(y\circ z)| = p^{2n}$ , since the $p^n+1$ points of the line $l_{xy}$ are not contained into the set $x\circ(y\circ z)$.
Similarly, the set $(x\circ y)\circ z$ does not contain the points of the line $l_{xy}$, since

$$(x\circ y)\circ z = (l_{xy} - \{x\})\circ z = (x_1\circ z) \cup (x_2\circ z) \cup \ldots\ldots \cup (x_{p^n-1}\circ z) \cup (y\circ z),$$

where $x_1, x_2, \ldots\ldots, x_{p^n-1} \in l_{xy}$.

Also, the set $(x\circ y)\circ z$ does not contain the points of the line $l_{xz}$, since $x\circ y = l_{xy} - \{x\}$.
As the lines $l_{xy}$ and $l_{xz}$ intersect at the point $x$, they don't have any other points in common. That means that the $p^{2n}-p^n+1$ points of the set $(x\circ y)\circ z$ (proposition 7), are also points of the set $x\circ(y\circ z)$.
So, we proved that $(x\circ y)\circ z \subset x\circ(y\circ z)$, which, generally, means that

$$(x\circ y)\circ z \cap x\circ(y\circ z) \neq \varnothing \quad \text{for every } x,y,z \in V. \ \blacksquare$$

**Remark 9.** For the $H_v$-group $(V,\circ)$, since $y \in x\circ y$ $\forall x,y \in V$, the $V/\beta^*$ is a singleton.

Since, $y \in x\circ y$ for every $x,y \in V$, we get the following proposition:

**Proposition 10.** Every element of the $H_v$-commutative group $(V,\circ)$ is simultaneously a right zero and a left unit element.

**Example 11.** By means of the marks of a Galois Field, we shall now give a concrete representation of a finite *2*-dimensional projective geometry.
We denote a point of the geometry by the ordered set of coordinates $(\mu_0, \mu_1, \mu_2)$. The $\mu_0, \mu_1, \mu_2$ are marks of the $GF[2^2]$ defined by means of the function $x^2+x+1$. At least one of $\mu_0, \mu_1, \mu_2$ is different from

zero. Let us denote the marks of the GF[$2^2$] by 0, 1, a, b, then we have the following tables :

| + | **0** | **1** | **a** | **b** |
|---|---|---|---|---|
| **0** | 0 | 1 | a | b |
| **1** | 1 | 0 | b | a |
| **a** | a | b | 0 | 1 |
| **b** | b | a | 1 | 0 |

| · | **0** | **1** | **a** | **b** |
|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 |
| **1** | 0 | 1 | a | b |
| **a** | 0 | a | b | 1 |
| **b** | 0 | b | 1 | a |

The ordered set of marks $\mu_0$, $\mu_1$, $\mu_2$ may be chosen by $(2^2)^{2+1} = 64$ ways, but since the symbol (0, 0, 0) is excepted, it may be chosen by 64-1=63 ways. So, there exist 63 points. In this totality, each point is represented by 3 ways (3 sets of symbols, since there are 3 nonzero marks in the field). Then the number of points defined is $63 \div 3 = 21$.
This representation of the finite *2*-dimensional projective geometry by means of the marks of the GF[$2^2$], constitute the projective geometry PG(2, $2^2$).
The 21 points of PG(2, $2^2$) = V will be denoted by letters in accordance with the following scheme :

A(001) B(010) C(011) D(01a) E(01b) F(100) G(101)

H(10a) I(10b) J(110) K(111) L(11a) M(11b) N(1a0)

O(1a1) P(1aa) Q(1ab) R(1b0) S(1b1) T(1ba) U(1bb)

Now, for the line containing the two distinct points A(001) and B(010) we take the set of points :

$$( \mu 0 + \nu 0 , \mu 0 + \nu 1 , \mu 1 + \nu 0 )$$

where $\mu$ and $\nu$ run independently over the marks of the GF[$2^2$] subjected to the condition that $\mu$ and $\nu$ shall not be simultaneously zero.
Then the number of possible combinations of the $\mu$ and $\nu$ is $(2^2)^2 - 1 = 15$. For each of these combinations, the corresponding symbol

denotes a point. But the same point is represented by 3 of these combinations of $\mu$ and $\nu$, due to the factor of proportionality involved in the definition of points. So, we get the following scheme:

A : **(001)** , (00a) , (00b)
B : **(010)** , (0a0) , (0b0)
C : **(011)** , (0aa) , (0bb)
D : **(01a)** , (0b1) , (0ab)
E : **(01b)** , (0ba) , (0a1)

Therefore, a line so defined, contains the 15 ÷ 3 = 5 points A,B,C,D,E. It is obvious that any two points on the line may be used in this way to define the same line.
The 21 lines are those given in the following scheme and the letters in a given column denoting a line:

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | A | A | A | A | B | B | B | B | C | C | C | C | D | D | D | D | E | E | E | E |
| B | F | J | N | R | F | G | H | I | F | G | H | I | F | G | H | I | F | G | H | I |
| C | G | K | O | S | J | K | L | M | K | J | M | L | L | M | J | K | M | L | K | J |
| D | H | L | P | T | N | O | P | Q | P | Q | N | O | Q | P | O | N | O | N | Q | P |
| E | I | M | Q | U | R | S | T | U | U | T | S | R | S | R | U | T | T | U | R | S |

Let us take the noncolliner points A, B, F, then

$$A\circ(B\circ F) = A\circ\{F, J, N, R\} =$$
$$=\{F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U\}.$$
$$(A\circ B)\circ F = \{B, C, D, E\}\circ F = \{F, J, N, R, K, P, U, L, Q, S, M, O, T\}.$$

Then, it follows that $(A\circ B)\circ F \subset A\circ(B\circ F)$.

Also, for the hyperoperation $(\cdot)$ we proved that $|x\cdot(y\cdot z)| = |(x\cdot y)\cdot z| = p^{2n} + p^n + 1$ and since the set $V = PG(2, 2^2)$ consists of 21 points, it follows that

$$x\cdot(y\cdot z) = (x\cdot y)\cdot z = V \quad \text{for every } x,y,z\in V.$$

## 5. On a dual $H_v$-ring of finite order

Working on dual $H_v$-rings $(H,+,\cdot)$, one needs to prove not only the weak distributivity of $(\cdot)$ with respect to $(+)$ but also the weak distributivity of $(+)$ with respect to $(\cdot)$.

Since the set V is now equipped with the hyperoperations $(\cdot)$ and $(\circ)$ mentioned above, the next propositions 12 to 19 serve the above purpose.
Similarly, as in proposition 2, we can prove that:

**Proposition 12.** For every noncollinear $x,y,z\in V$

$$|(x\cdot y)\cdot(x\cdot z)| = |(x\cdot z)\cdot(y\cdot z)| = p^{2n}+p^{n}+1$$

Following a similar procedure, as in proposition 3 and according to the propositions 2 and 12, we get the next proposition:

**Proposition 13.** For every $x,y,z\in V$

$$x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z) \quad \text{and} \quad (x\cdot y)\cdot z = (x\cdot z)\cdot(y\cdot z)$$

**Proposition 14.** For every noncolliner $x,y,z\in V$,

$$|x\cdot(y\circ z)| = p^{2n}+1 \quad \text{and} \quad |(x\cdot y)\circ(x\cdot z)| = p^{2n}+p^{n}+1$$

**Proof.** First we consider the set $x\cdot(y\circ z)$. Since the point y does not belong to the line $y\circ z$, we get that $|y\circ z| = p^n$. Also, for every $w\in y\circ z$ we get that $|x\cdot w|=p^n+1$. Since the point x appears $p^n$ times in the set $x\cdot(y\circ z)$, we have

$$|x\cdot(y\circ z)|=(p^n+1)\,p^n - p^n+1 = p^{2n}+1.$$

Consider now the set $(x\cdot y)\circ(x\cdot z)$. Each of the lines $x\cdot y$ and $x\cdot z$ contains $p^n+1$ points, having in common only the point x, since the points x,y,z are noncollinear. Then, we get the following:

i) $x\circ x = x$, by definition.

ii) Due to the hyperoperation $(\circ)$, the point $x\in x\cdot y$ together with the rest $p^n$ points of the line $x\cdot z$ create the $p^n$ points of the line $x\cdot z$.

iii) Due to the hyperoperation $(\circ)$, the point $y\in x\cdot y$ together with the $p^n+1$ points of the line $x\cdot z$, create $(p^n+1)-2 = p^n-1$ points each time. Indeed, the point y does not participate (by definition) and the point $w\in x\cdot z$ (which appears due to the hyperoperation $y\circ w$), already exists due to the hyperoperation $x\circ w$ of the case (ii).

iv) Also, the point $y \in (x \cdot y) \circ (x \cdot z)$. Indeed, since there exists $w' \in x \cdot y$ such that $y \in w' \circ x$ (where $x \in x \cdot z$).

So, from the above 4 cases we get that:

$$|(x \cdot y) \circ (x \cdot z)| = 1 + p^n + (p^n - 1)(p^n + 1) + 1 = p^{2n} + p^n + 1. \blacksquare$$

In a similar way, we get the following proposition:

**Proposition 15.** For every noncolliner $x,y,z \in V$

$$|(x \circ y) \cdot z| = p^{2n} + 1 \quad \text{and} \quad |(x \cdot z) \circ (y \cdot z)| = p^{2n} + p^n + 1$$

**Proposition 16.** For every noncolliner $x,y,z \in V$,

$$|x \circ (y \cdot z)| = |(x \circ y) \cdot (x \circ z)| = p^{2n} + p^n$$

**Proof.** First, consider the set $x \circ (y \cdot z)$. The line $y \cdot z$ consists of $p^n + 1$ points. The point $x$ (due to the hyperoperation $(\circ)$) together with the points of the line $y \cdot z$ creates each time $(p^n + 1)\, p^n = p^{2n} + p^n$ points, since $x$ is not participating.

Consider now, the set $(x \circ y) \cdot (x \circ z)$. Each of the lines $x \circ y$ and $x \circ z$ contains $p^n$ different points, since the point $x$ is not participating. Then, we get the following:

i) The point $y \in x \circ y$ (due to the hyperoperation $(\cdot)$), together with every $w \in x \circ z$ creates $p^n + 1$ points each time, but since the point y appears $p^n$ times we get that the number of points in this case is $(p^n + 1)\, p^n - p^n + 1 = p^{2n} + 1$.

ii) The point $w' \in (x \circ y) - \{y\}$ (due to the hyperoperation $(\cdot)$), together with the point $w \in x \circ z$ creates each time the point $w'$, since the point $w$ already exists from case (i). The number of those $w'$'s is $p^n - 1$.

Then, the set $(x \circ y) \cdot (x \circ z)$ consists of $(p^{2n} + 1) + (p^n - 1) = p^{2n} + p^n$ points. As we mentioned, $x$ is the only point which is not participating. $\blacksquare$

Similarly, we get the following two propositions:

**Proposition 17.** For every noncolliner $x,y,z \in V$

$$|(x\cdot y)\circ z| = p^{2n} \quad \text{and} \quad |(x\circ z)\cdot(y\circ z)| = p^{2n}+p^{n}+1$$

**Proposition 18.** For every noncollinear $x,y,z\in V$

$$|(x\circ y)\circ(x\circ z)| = p^{2n} \quad \text{and} \quad |(x\circ z)\circ(y\circ z)| = p^{2n}+p^{n}+1$$

Following a similar procedure as above and according to the proposition 18 we get the next proposition:

**Proposition 19.** For every $x,y,z\in V$

$$x\circ(y\circ z) = (x\circ y)\circ(x\circ z) \quad \text{and} \quad (x\circ y)\circ z \subset (x\circ z)\circ(y\circ z)$$

**Proposition 20.** The hyperstructure $(V,▫,▪)$, where $▫,▪\in\{\cdot,\circ\}$, is a dual $H_v$-ring.

**Proof.** There are four hyperstructures: $(V,\cdot,\cdot)$ , $(V,\circ,\circ)$ , $(V,\cdot,\circ)$ , $(V,\circ,\cdot)$.
The hyperoperations $(\cdot)$, $(\circ)$ (by propositions 3 and 8 respectively) are satisfying the reproduction axiom.
The hyperoperation $(\cdot)$ (by proposition 3) is associative and the hyperoperation $(\circ)$ (by proposition 8) is weak associative.
Now, for the distributivity or the weak distributivity of (▪) with respect to (▫) we have the following cases:
By proposition 13:

$$x\cdot(y\cdot z) = (x\cdot y)\cdot(x\cdot z) \quad \text{and} \quad (x\cdot y)\cdot z = (x\cdot z)\cdot(y\cdot z) \text{ for every } x,y,z\in V.$$

By proposition 19:

$$x\circ(y\circ z) = (x\circ y)\circ(x\circ z) \quad \text{and} \quad (x\circ y)\circ z \cap (x\circ z)\circ(y\circ z)\neq\varnothing \quad \forall\ x,y,z\in V.$$

Following a similar procedure as for the distributivity of the above hyperoperations and taking into account the propositions 16, 17, 18, 19 we get that:

$$x\circ(y\cdot z)=(x\circ y)\cdot(x\circ z) \quad \text{for every } x,y,z\in V.$$

On the right side, $(x\cdot y)\circ z \subset (x\circ z)\cdot(y\circ z)$ is valid, which means that

$$(x\cdot y)\circ z \cap (x\circ z)\cdot(y\circ z)\neq\varnothing \quad \text{for every } x,y,z\in V.$$

Also, $x\cdot(y\circ z) \subset (x\cdot y)\circ(x\cdot z)$, which means that

$$x\cdot(y\circ z) \cap (x\cdot y)\circ(x\cdot z) \neq \varnothing \quad \text{for every } x,y,z\in V.$$

Finally, on the right hand side $(x\circ y)\cdot z \subset (x\cdot z)\circ(y\cdot z)$, which means that

$$(x\circ y)\cdot z \cap (x\cdot z)\circ(y\cdot z) \neq \varnothing \quad \text{for every } x,y,z\in V.$$

So, the hyperstructure $(V,\square,\blacksquare)$, where $\square,\blacksquare\in\{\cdot,\circ\}$, is a dual $H_v$-ring.■

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