

# Common Neighbor Polynomial of Some Special Trees

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## Abstract

Let  $G(V, E)$  be a simple graph of order  $n$  with vertex set  $V$  and edge set  $E$ . Let  $(u, v)$  denotes an unordered vertex pair of distinct vertices of  $G$  and let  $N(u)$  denote the open neighborhood of the vertex  $u$  in  $G$ . The  $i$ -common neighbor set of  $G$  is defined as  $N(G, i) := \{(u, v) : u, v \in V, u \neq v \text{ and } |N(u) \cap N(v)| = i\}$ , for  $0 \leq i \leq n - 2$ . The polynomial  $N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$  is defined as the common neighbor polynomial of  $G$ . In this paper, we study the common neighbor polynomial of some special type of trees such as complete  $m$ -ary tree, caterpillar tree, star like tree and fire cracker graph.

**Keywords:** common neighbor, common neighbor polynomial.

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## 1 Introduction

Vertex similarity of nodes is a well studied concept in graph theory as it is highly significant in various fields such as in the study of molecular structure of chemical graphs, measuring consensus rate of different individuals/organizations in network analysis etc. The number of common neighbors shared by two nodes in a network system can be treated as a measure of consensus among the corresponding individuals. This motivates the authors to define the  $i$ -common neighbor set and the common neighbor polynomial of a graph.

Let  $G(V, E)$  be a graph (simple graph) of order  $n$  with vertex set  $V$  and edge set  $E$ . Let  $(u, v)$  denotes an unordered vertex pair of distinct vertices of  $G$  and let  $N(u)$  denote the open neighborhood of the vertex  $u$  in  $G$ . The  $i$ -common neighbor set of  $G$  is defined as  $N(G, i) := \{(u, v) : u, v \in V, u \neq v \text{ and } |N(u) \cap N(v)| = i\}$ , for  $0 \leq i \leq n - 2$ . The polynomial  $N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$  is defined as the common neighbor polynomial of  $G$ . This polynomial was introduced by the present authors in (3).

Trees are commonly used to represent hierarchical data structures involved in network file system, possibility spaces, algorithmic routing etc. It is easy to dissect a tree data structure to access the information about a particular part of a huge data. Due to this reason, many researches were conducted to explore the properties of trees. In this paper, we derive the common neighbor polynomial of some special trees like rooted trees, caterpillars etc.

## 2 Main Results

A rooted tree is a tree in which one of the vertices is distinguished as the root. According to the distance of other vertices from the root vertex, there is a hierarchy on the vertices of a rooted tree. The distance of a vertex  $v$  from the root is called the depth or level of the vertex. The height of a rooted tree is the greatest depth of a vertex of the tree. Considering a path from the root to a vertex  $w$ , if a vertex  $v$  immediately precedes  $w$ , then  $v$  is called the parent of  $w$  and  $w$  is called the child of  $v$ . Vertices having same parent are called siblings. An  $m$ -ary tree ( $m \geq 2$ ) is a rooted tree in which every vertex has  $m$  or fewer number of children. A complete  $m$ -ary tree is an  $m$ -ary tree in which every internal vertices has exactly  $m$  children and all leaves are of same distance from the root.

**Theorem 1.** *Let  $T$  be a tree on  $n$  vertices. Let  $v$  be a vertex of  $T$  with degree  $k$ . If  $T'$  is a tree obtained from  $T$  by attaching  $p$  pendent edges at the vertex  $v$ , we have the following:*

$$N[T'; x] = N[T; x] + \frac{p}{2}(2k + p - 1)x + p(n - k).$$

*Common neighbor polynomial of some special trees*

*Proof.* Let  $\{v_1, v_2, \dots, v_k\}$  be the neighbors of  $v$  in  $T$ . When we attach a pendent edge  $vw$  to  $v$ ,  $n$  number of new pairs of vertices are introduced in which the pairs  $(v_i, w)$  where  $i \in \{1, 2, \dots, k\}$  have one common neighbor  $v$  and the remaining  $n - k$  new pairs have no common neighbors. There will be no change in the number of common neighbors of pairs of vertices of  $T$  by the introduction of the pendent edge  $vw$ . Hence the common neighbor polynomial becomes  $N[T; x] + kx + (n - k)$ . Repeating the process  $p$  times, after attaching the  $p$ -th pendent edge to  $v$ , the common neighbor polynomial of resulting graph becomes,

$$\begin{aligned} N[T'; x] &= N[T; x] + kx + (n - k) + (k + 1)x + (n - k) + \dots \\ &\quad + (k + p - 1)x + (n - k) \\ &= N[T; x] + [k + (k + 1) + (k + 2) + \dots + (k + p - 1)]x + p(n - k) \\ &= N[T; x] + \frac{p}{2}(2k + p - 1)x + p(n - k). \end{aligned}$$

This completes the proof. □

**Theorem 2.** *Let  $T$  be a complete  $m$ -ary tree with  $p$  levels where the root vertex is considered to be in the 0-th level. Then we have the following:*

$$\begin{aligned} N[T; x] &= \frac{m^2(m^{p-1} - 1)}{m - 1}x + \binom{m}{2} \frac{m^p - 1}{m - 1}x + \frac{m[m^{2p-2} - m^p + m - 1]}{m - 1} + m^{2p-1} \\ &\quad + \sum_{i=0}^{p-3} \left[ \frac{m^{2i+3} - m^{p+i+1}}{1 - m} \right] + \frac{m^2[m^{2p} - m^{p+1} - m^p + m]}{2(m^2 - 1)}. \end{aligned}$$

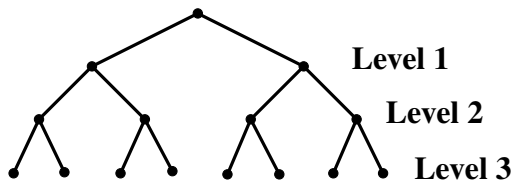


Figure 1: Complete binary tree with 3 levels of vertices

*Proof.* Let  $(u, v)$  be any pair of vertices of  $T$ . Here we consider 4 different cases according to the levels in which the vertices  $u$  and  $v$  lie in  $T$ .

**Case(i)** For  $i \in \{0, 1, 2, \dots, p - 2\}$  let  $u$  be a vertex in the  $i$ -th level and  $v$  a vertex in the  $(i + 1)^{th}$  or  $(i + 2)^{th}$  level.

If  $v$  is an  $(i + 1)^{th}$  level vertex, the vertex pair  $(u, v)$  have no common neighbors and there are  $m^i m^{i+1}$  such pairs of vertices in  $T$ . If  $v$  is in the  $(i + 2)^{th}$

level, then there are  $m^i m^{i+2}$  pairs of vertices  $(u, v)$  in which  $m^i[m^{i+2} - m^2]$  pairs of vertices have no common neighbors and  $m^i m^2$  pairs have exactly one common neighbor.

**Case(ii)** Let  $u$  be a vertex in the  $(p - 1)$ -th level and  $v$  a vertex in the  $p$ -th level. In this case the pairs of vertices  $(u, v)$  have no common neighbors and there are  $m^{p-1} m^p$  such pairs of vertices.

**Case(iii)** For  $i \in \{0, 1, 2, \dots, p - 3\}$  let  $u$  be a vertex in the  $i$ -th level and  $v$  a vertex in the  $j$ -th level where  $j = i + 3, i + 4, \dots, p$ . All the pairs of vertices under this case have no common neighbors and there are  $m^i[m^{i+3} + m^{i+4} + \dots + m^p]$  such pairs of vertices.

**Case(iv)** For  $i \in \{1, 2, \dots, p\}$  let  $u$  and  $v$  be vertices of same level. In this case  $\frac{1}{2}m^i[m^i - m]$  distinct pairs of vertices which are not siblings have no common neighbors and  $\binom{m}{2}m^{i-1}$  pairs of vertices which are siblings have exactly one common neighbor.

From the above cases, it follows that

$$\begin{aligned} N[T; x] &= \sum_{i=0}^{p-2} m^i m^2 x + \binom{m}{2} \sum_{i=1}^p m^{i-1} x + \sum_{i=0}^{p-2} m^i [m^{i+1} + m^{i+2} - m^2] + m^{2p-1} \\ &\quad + \sum_{i=0}^{p-3} m^i [m^{i+3} + m^{i+4} + \dots + m^p] + \sum_{i=1}^p \frac{1}{2} m^i (m^i - m). \\ &= \frac{m^2(m^{p-1} - 1)}{m - 1} x + \binom{m}{2} \frac{m^p - 1}{m - 1} x + \frac{m[m^{2p-2} - m^p + m - 1]}{m - 1} + m^{2p-1} \\ &\quad + \sum_{i=0}^{p-3} \left[ \frac{m^{p+i+1} - m^{2i+3}}{m - 1} \right] + \frac{m^2[m^{2p} - m^{p+1} - m^p + m]}{2(m^2 - 1)}. \end{aligned}$$

This completes the proof. □

A derivative of a graph  $G$  is a graph obtained from  $G$  by deleting all the pendent vertices of  $G$ . A caterpillar is a tree whose derivative is a path graph (5). Consequently, a caterpillar  $P_n(m_1, m_2, \dots, m_n)$  is obtained by attaching  $m_i$  pendent edges to the vertex  $v_i$  of a path  $P_n$  where  $i \in \{1, 2, \dots, n\}$ .

**Theorem 3.** *The common neighbor polynomial of a caterpillar tree  $P_n(m_1, m_2, \dots, m_n)$  is given by the following:*

Common neighbor polynomial of some special trees

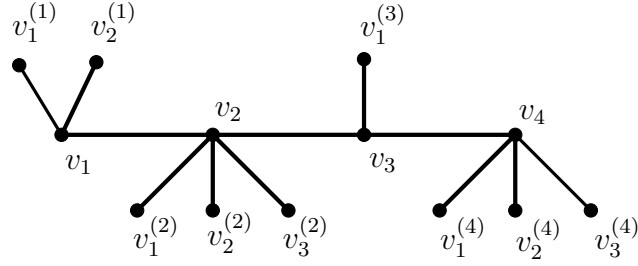


Figure 2: The caterpillar  $P_4(2, 3, 1, 3)$

$$N[P_n(m_1, m_2, \dots, m_n); x] = N[P_n; x] + \sum_{j=1}^n N[K_{1, m_j}; x] + \sum_{\substack{l, k \in \{1, 2, \dots, n\} \\ l \neq k}} m_l m_k \\ + \left[ m_1 + m_n + 2 \sum_{j=2}^{n-1} m_j \right] x + (n-2)(m_1 + m_n) + (n-3) \sum_{j=2}^{n-1} m_j.$$

*Proof.* Let  $P_n(m_1, m_2, \dots, m_n)$  be a caterpillar and let  $v_1, v_2, \dots, v_n$  be the vertices of its derived graph which is a path. Also let  $v_1^{(j)}, v_2^{(j)}, \dots, v_{m_j}^{(j)}$  be the pendent vertices of the caterpillar tree attached to the vertex  $v_j$  where  $j \in \{1, 2, \dots, n\}$ . Let  $(u, v)$  be any pair of vertices of the caterpillar. We consider the following cases to build up its common neighbor polynomial.

**Case(i)** Let  $u, v \in \{v_1, v_2, \dots, v_n\}$ . Here the number of pairs of vertices  $(u, v)$  with  $i$  common neighbors equals  $|N(P_n, i)|$ . So the pairs of vertices under this case contribute the term  $N[P_n; x]$  to the common neighbor polynomial of the caterpillar.

**Case(ii)** Let  $u, v \in \{v_j, v_1^{(j)}, v_2^{(j)}, \dots, v_{m_j}^{(j)}\}$ . Here the vertices of the set under consideration spans a star graph  $K_{1, m_j}$  and hence the number of pairs of vertices  $(u, v)$  with  $i$  common neighbors equals  $|N(K_{1, m_j}, i)|$ .

**Case(iii)** Let  $u \in \{v_1^{(l)}, v_2^{(l)}, \dots, v_{m_l}^{(l)}\}$  and  $v \in \{v_1^{(k)}, v_2^{(k)}, \dots, v_{m_k}^{(k)}\}$  where  $l \neq k$  and  $l, k \in \{1, 2, \dots, n\}$ .

Here  $u$  and  $v$  are the pendent vertices attached to the vertices  $v_l$  and  $v_k$  respectively where  $l \neq k$ . No pair of vertices under this case have common neighbors and there are  $\sum_{l \neq k} m_l m_k$  such pairs.

**Case(iv)** Let  $u \in \{v_1, v_n\}$ , a pendent vertex of the derived graph  $P_n$  and let  $v$  be any vertex attached to the vertices of  $P_n$  such that  $uv$  is not an edge of the caterpillar.

In this case pairs of vertices of the form  $(v_1, v_l^{(2)})$  and  $(v_n, v_k^{(n-1)})$  where  $l \in \{1, 2, \dots, m_2\}$  and  $k \in \{1, 2, \dots, m_{n-1}\}$  have exactly one common neighbor each and there are  $m_2 + m_{n-1}$  such pairs of vertices. There remains  $m_1 + m_2 + m_{n-1} + m_n + 2 \sum_{j=3}^{n-2} m_j$  pairs of vertices under this case which have no common neighbors.

**Case(v)** Let  $u \in \{v_2, v_3, \dots, v_{n-1}\}$  and let  $v$  be any vertex selected in a way same as in Case(iv).

For  $i \in \{2, 3, \dots, n-1\}$ , the pairs of vertices of the form  $(v_i, v_l^{(i-1)})$  where  $l \in \{1, 2, \dots, m_{i-1}\}$  have exactly one common neighbor  $v_{i-1}$  and pairs of vertices of the form  $(v_i, v_k^{(i+1)})$  where  $k \in \{1, 2, \dots, m_{i+1}\}$  have exactly one common neighbor  $v_{i+1}$ . There are  $m_1 + m_2 + m_{n-1} + m_n + 2 \sum_{j=3}^{n-2} m_j$  such pairs of vertices.

The remaining pairs of vertices under this case have no common neighbors and there are  $(n-3)(m_1 + m_n) + (n-4)(m_2 + m_{n-1}) + (n-5) \sum_{j=3}^{n-2} m_j$  such pairs of vertices.

From the above cases, it follows that

$$\begin{aligned} N[P_n(m_1, m_2, \dots, m_n); x] &= N[P_n; x] + \sum_{j=1}^n N[K_{1, m_j}; x] + \sum_{\substack{l, k \\ l \neq k}} m_l m_k \\ &+ [m_2 + m_{n-1}]x + m_1 + m_2 + m_{n-1} + m_n + 2 \sum_{j=3}^{n-2} m_j \\ &+ \left[ m_1 + m_2 + m_{n-1} + m_n + 2 \sum_{j=3}^{n-2} m_j \right] x + (n-3)(m_1 + m_n) \\ &+ (n-4)(m_2 + m_{n-1}) + (n-5) \sum_{j=3}^{n-2} m_j. \end{aligned}$$

Now the result follows after some rearrangement of the terms. □

**Corollary 4.** For a caterpillar  $P_n(m, m, \dots, m)$  where  $m$  pendent edges are attached to each vertex of the path  $P_n$ , we have

$$\begin{aligned} N[P_n(m, m, \dots, m); x] &= N[P_n; x] + nN[K_{1, m}; x] + \binom{n}{2} m^2 + 2m(n-1)x \\ &+ m(n-1)(n-2). \end{aligned}$$

A star like tree(2)  $S(n_1, n_2, \dots, n_k)$  is a graph having only one vertex  $w$  of degree greater than 2 such that deletion of  $w$  results in a disjoint union of the path

*Common neighbor polynomial of some special trees*

graphs  $P_{n_1}, P_{n_2}, \dots, P_{n_k}$ . The star like tree graphs are used to represent proteins which will have generally 20 branches where each branch indicates the presence of one of the 20 natural amino acids. For example, the strand A of human insulin has 21 amino acids of 11 kinds which can be modelled by a star like tree graph with 11 branches as shown in Figure 3.

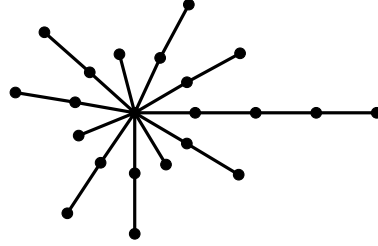


Figure 3: Strand A of human insulin

**Theorem 5.** *The common neighbor polynomial of a star like tree  $S(n_1, n_2, \dots, n_k)$  with  $N + 1$  vertices is given by*

$$N[S(n_1, n_2, \dots, n_k); x] = \sum_{r=1}^k N[P_{n_r+1}; x] + \binom{k}{2}x + \binom{N}{2} - \sum_{r=1}^k \binom{n_r}{2} - \binom{k}{2}$$

where  $N = n_1 + n_2 + \dots + n_k$ .

*Proof.* Let  $S(n_1, n_2, \dots, n_k)$  be a star like tree with a vertex  $w$  such that  $S(n_1, n_2, \dots, n_k) - w = P_{n_1} \cup P_{n_2} \cup \dots \cup P_{n_k}$ . Any pair of vertices  $(u, v) \in P_{n_r} \cup \{w\}$  where  $r \in \{1, 2, \dots, k\}$ , has as many common neighbors in  $S(n_1, n_2, \dots, n_k)$  as it has in  $P_{n_r+1}$ .

Let  $(u, v)$  be a pair of vertices in  $S(n_1, n_2, \dots, n_k)$  such that  $u \in P_{n_r}$  and  $v \in P_{n_s}$  where  $r \neq s, r, s \in \{1, 2, \dots, k\}$ . Then the vertex pair  $(u, v)$  has a single common neighbor  $w$  if both  $u$  and  $v$  are adjacent to  $w$  and there are no common neighbors otherwise. Hence there are  $\binom{k}{2}$  pairs of vertices with one common neighbor and  $\binom{N}{2} - \sum_{r=1}^k \binom{n_r}{2} - \binom{k}{2}$  pairs of vertices with no common neighbors. Hence the result follows.  $\square$

**Corollary 6.** *The common neighbor polynomial of the graphical representation of Strand A of human insulin is given by*

$$N[G; x] = 65x + 166.$$

*Proof.* The proof follows from the fact that Strand A of human insulin can be graphically represented as a star like tree graph  $S(1, 2, 4, 2, 2, 1, 2, 2, 1, 2, 2)$ .  $\square$

The  $(n, k)$  firecracker graph(1) is obtained by identifying each vertex of a path  $P_n$  with one of the pendent vertices of the star graph  $K_{1,k}$ . In particular, the  $(n, 2)$  firecracker graph is known as the centipede graph.

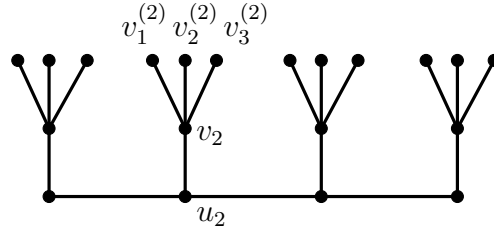


Figure 4: The  $(4, 4)$ - firecracker graph

**Lemma 7.** (see (3)) For a complete bipartite graph  $K_{m,n}$ , we have

$$N[K_{m,n}; x] = \binom{m}{2}x^n + \binom{n}{2}x^m + mn.$$

**Lemma 8.** (see (3)) For a path  $P_n$  with  $n \geq 2$  vertices, we have

$$N[P_n; x] = (n - 2)x + \binom{n - 1}{2} + 1.$$

**Theorem 9.** The common neighbor polynomial of  $(n, k)$  firecracker graph  $G$  is given by the following:

$$N[G; x] = \left[ n \binom{k}{2} + 3n - 4 \right] x + nk + \binom{n-1}{2} + 1 + \binom{n}{2}k^2 + (n - 1)(n - 2) + n(n - 1)(k - 1).$$

*Proof.* For  $j \in \{1, 2, \dots, n\}$ , let  $v_1^j, v_2^j, \dots, v_k^j$  be the pendent vertices of the  $j^{th}$  star where the vertex  $v_k^j$  is identified with the vertex  $u_j$  of the path  $P_n$ . Let  $v_j$  be the center vertex of the  $j^{th}$  star attached to the vertex  $u_j$  of  $P_n$ . Let  $(u, v)$  be any pair of vertices of  $G$ . Here we consider 5 cases:

**Case(i)** Let  $u, v \in \{v_j, v_1^j, v_2^j, \dots, v_k^j\}$  where  $j \in \{1, 2, \dots, n\}$ .

In this case, the pair  $(u, v)$  has as many common neighbors in  $G$  as in  $K_{1,k}$ .

**Case(ii)** Let  $u, v \in \{u_1, u_2, \dots, u_n\}$ .

Here the vertex pair  $(u, v)$  has as many common neighbors in  $G$  as in  $P_n$ .

**Case(iii)** Let  $u = v_j$  and  $v \in \{u_1, u_2, \dots, u_{j-1}, u_{j+1}, \dots, u_n\}$  where  $j \in \{1, 2, \dots, n\}$

In this case, for each  $j$ , the pairs  $(v_j, u_{j-1})$  and  $(v_j, u_{j+1})$  has exactly one



*Common neighbor polynomial of some special trees*

common neighbor. Also the pairs  $(v_1, u_2)$  and  $(v_n, u_{n-1})$  have one common neighbor each. Hence there are  $2(n - 1)$  pairs of vertices  $(u, v)$  with one common neighbor and all other  $(n - 1)(n - 2)$  pairs of vertices have no common neighbors.

**Case(iv)** Let  $u \in \{v_r, v_1^r, v_2^r, \dots, v_{k-1}^r\}$  and  $v \in \{v_s, v_1^s, v_2^s, \dots, v_{k-1}^s\}$  where  $r, s \in \{1, 2, \dots, n\}$ .

In this case, there are  $\binom{n}{2}k^2$  pairs of vertices which have no common neighbors.

**Case(v)** Let  $u \in \{v_1^j, v_2^j, \dots, v_{k-1}^j\}$  and  $v \in \{u_1, u_2, \dots, u_{j-1}, u_{j+1}, \dots, u_n\}$  where  $j \in \{1, 2, \dots, n\}$ ,

In this case, there are  $n(n - 1)(k - 1)$  pairs of vertices which have no common neighbors.

Using lemmas 7 and 8, it follows that,

$$\begin{aligned} N[G; x] &= nN[K_{1,k}; x] + N[P_n; x] + 2(n - 1)x + (n - 1)(n - 2) + \binom{n}{2}k^2 \\ &\quad + n(n - 1)(k - 1) \\ &= n\left[\binom{k}{2}x + k\right] + (n - 2)x + \binom{n - 1}{2} + 1 + 2(n - 1)x + \binom{n}{2}k^2 \\ &\quad + (n - 1)(n - 2) + n(n - 1)(k - 1) \\ &= \left[n\binom{k}{2} + 3n - 4\right]x + nk + \binom{n - 1}{2} + 1 + \binom{n}{2}k^2 \\ &\quad + (n - 1)(n - 2) + n(n - 1)(k - 1). \end{aligned}$$

This completes the proof. □

**Corollary 10.** *The common neighbor polynomial of the centipede graph  $G$  is given by*

$$N[G; x] = 4(n - 1)x + 2n + \frac{3(n - 1)(3n - 2)}{2} + 1.$$

*Proof.* The proof follows from the fact that the centipede graph is a special case of  $(n, k)$ - firecracker graph when  $k = 2$ . □

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