Gamma Modules

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Abstract

Let R be a Γ-ring. We introduce the notion of gamma modules over R and study important properties of such modules. In this regards we study submodules and homomorphism of gamma modules and give related basic results of gamma modules.

Keywords: Γ-ring, R_Γ-module, Submodule, Homomorphism.

1 Introduction

The notion of a Γ-ring was introduced by N. Nobusawa in [6]. Recently, W.E. Barnes [2], J. Luh [5], W.E. Coppage studied the structure of Γ-rings and obtained various generalization analogous of corresponding parts in ring theory. In this paper we extend the concepts of module from the category of rings to the category of R_Γ-modules over Γ-rings. Indeed we show that the notion of a gamma module is a generalization of a Γ-ring as well as a module over a ring, in fact we show that many, but not all, of the results in the theory of modules are also valid for R_Γ-modules. In Section 2, some definitions and results of $\Gamma - ring$ which will be used in the sequel are given. In Section 3, the notion of a Γ-module M over a $\Gamma - ring$ R is given and by many example it is shown that the class of Γ-modules is very wide, in fact it is shown that the notion of a Γ-module is a generalization of an ordinary module and a $\Gamma - ring$. In Section 3, we study the submodules of a given Γ-module. In particular, we that $L(M)$, the set of all submodules of a Γ-module M constitute a complete lattice. In Section 3, homomorphisms of Γ-modules are studied and the well known homomorphisms (isomorphisms) theorems of modules extended for Γ-modules. Also, the behavior of Γ-submodules under homomorphisms are investigated.

2 Preliminaries

Recall that for additive abelian groups R and Γ we say that R is a *$\Gamma - ring$* if there exists a mapping

$$\begin{aligned} \cdot : R \times \Gamma \times R &\longrightarrow R \\ (r, \gamma, r') &\longmapsto \quad r\gamma r' \end{aligned}$$

such that for every $a, b, c \in R$ and $\alpha, \beta \in \Gamma$, the following hold:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$;

$a(\alpha + \beta)c = a\alpha c + a\beta c$;

$a\alpha(b + c) = a\alpha b + a\alpha c$;

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

A subset A of a Γ-ring R is said to be a *right ideal* of R if A is an additive subgroup of R and $A\Gamma R \subseteq A$, where $A\Gamma R = \{a\alpha c |\ a \in A, \alpha \in \Gamma, r \in R\}$.

A *left ideal* of R is defined in a similar way. If A is both right and left ideal, we say that A is an *ideal* of R.

If R and S are Γ-rings. A pair (θ, φ) of maps from R into S such that

i) $\theta(x + y) = \theta(x) + \theta(y)$;

ii) φ is an isomorphism on Γ;

iii) $\theta(x\gamma y) = \theta(x)\varphi(\gamma)\theta(y)$.

is called a *homomorphism* from R into S.

3 $\mathbf{R_\Gamma}$ -Modules

In this section we introduce and study the notion of modules over a fixed Γ-ring.

Definition 3.1. Let R be a Γ-ring. A (left) R_Γ-module is an additive abelian group M together with a mapping $. : R \times \Gamma \times M \longrightarrow M$ (the image of (r, γ, m) being denoted by $r\gamma m$), such that for all $m, m_1, m_2 \in M$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$, $r, r_1, r_2 \in R$ the following hold:

(M_1) $\quad r\gamma(m_1 + m_2) = r\gamma m_1 + r\gamma m_2$;

(M_2) $\quad (r_1 + r_2)\gamma m = r_1\gamma m + r_2\gamma m$;

(M_3) $\quad r(\gamma_1 + \gamma_2)m = r\gamma_1 m + r\gamma_2 m$;

(M_4) $\quad r_1\gamma_1(r_2\gamma_2 m) = (r_1\gamma_1 r_2)\gamma_2 m$.

A *right* $R_\Gamma - module$ is defined in analogous manner.

Definition 3.2. A (left) R_Γ-module M is *unitary* if there exist elements, say 1 in R and $\gamma_0 \in \Gamma$, such that, $1\gamma_0 m = m$ for every $m \in M$. We denote $1\gamma_0$ by 1_{γ_0}, so $1_{\gamma_0}m = m$ for all $m \in M$.

Remark 3.3. If M is a left R_Γ-module then it is easy to verify that $0\gamma m = r0m = r\gamma 0 = 0_M$. If R and S are Γ-rings then an $(R, S)_\Gamma$-bimodule M is both a left R_Γ-module and right S_Γ-module and simultaneously such that $(r\alpha m)\beta s = r\alpha(m\beta s) \quad \forall m \in M, \forall r \in R, \forall s \in S$ and $\alpha, \beta \in \Gamma$.

In the following by many examples we illustrate the notion of gamma modules and show that the class of gamma module is very wide.

Example 3.4. If R is a Γ-ring, then every abelian group M can be made into an R_Γ-module with trivial module structure by defining

$r\gamma m = 0 \quad \forall r \in R, \forall \gamma \in \Gamma, \forall m \in M.$

Example 3.5. Every Γ-ring R, is an R_Γ-module with $r\gamma(r, s \in R, \gamma \in \Gamma)$ being the Γ-ring structure in R, i.e. the mapping

$$. : R \times \Gamma \times R \longrightarrow R.$$

$$(r, \gamma, s) \longmapsto r.\gamma.s$$

Example 3.6. Let M be a module over a ring A. Define $. : A \times R \times M \longrightarrow M$, by $(a, s, m) = (as)m$, being the R-module structure of M. Then M is an A_A-module.

Example 3.7. Let M be an arbitrary abelian group and S be an arbitrary subring of $\mathbb{Z}$, the ring of integers. Then M is a $\mathbb{Z}_S$-module under the mapping

$$. : \mathbb{Z} \times S \times M \longrightarrow M$$

$$(n, n', x) \longmapsto nn'x$$

Example 3.8. If R is a Γ-ring and I is a left ideal of R .Then I is an R_Γ-module under the mapping $. : R \times \Gamma \times I \longrightarrow I$ such that $(r, \gamma, a) \longmapsto r\gamma a$.

Example 3.9. Let R be an arbitrary commutative Γ-ring with identity. A polynomial in one indeterminate with coefficients in R is to be an expression $P(X) = a_n X^n + + a_2 X^2 + a_1 X + a_0$ in which X is a symbol, not a variable and the set $R[x]$ of all polynomials is then an abelian group. Now $R[x]$ becomes to an R_Γ-module, under the mapping

$$. : R \times \Gamma \times R[x] \longrightarrow R[x]$$

$$(r, \gamma, f(x)) \longmapsto r.\gamma.f(x) = \textstyle\sum_{i=1}^{n} (r\gamma a_i) x^i.$$

Example 3.10. If R is a Γ-ring and M is an R_Γ-module. Set $M[x] = \{\sum_{i=0}^{n} a_i x^i \mid a_i \in M\}$. For $f(x) = \sum_{j=0}^{n} b_j x^j$ and $g(x) = \sum_{i=0}^{m} a_i x^i$, define the mapping

$$. : R[x] \times \Gamma \times M[x] \longrightarrow M[x]$$
$$(g(x), \gamma, f(x)) \longmapsto g(x)\gamma f(x) = \sum_{k=1}^{m+n} (a_k.\gamma.b_k) x^k.$$

It is easy to verify that $M[x]$ is an $R[x]_\Gamma$-module.

Example 3.11. Let I be an ideal of a Γ-ring R. Then R/I is an R_Γ-module, where the mapping $. : R \times \Gamma \times R/I \longrightarrow R/I$ is defined by $(r, \gamma, r' + I) \longmapsto (r\gamma r') + I$.

Example 3.12. Let M be an R_Γ-module, $m \in M$. Letting $T(m) = \{t \in R \mid t\gamma m = 0 \ \forall \gamma \in \Gamma\}$. Then $T(m)$ is an R_Γ-module.

Proposition 3.12. Let R be a Γ-ring and $(M, +, .)$ be an R_Γ-module. Set $Sub(M) = \{X \mid X \subseteq M\}$, Then $\text{sub}(M)$ is an R_Γ-module.

proof. Define $\oplus : (A, B) \longmapsto A \oplus B$ by $A \oplus B = (A \backslash B) \cup (B \backslash A)$ for $A, B \in sub(M)$. Then $(Sub(M), \oplus)$ is an additive group with identity element $\emptyset$ and the inverse of each element A is itself. Consider the mapping:

$$\circ : R \times \Gamma \times Sub(M) \longrightarrow sub(M)$$
$$(r, \gamma, X) \longmapsto r \circ \gamma \circ X = r\gamma X,$$

where $r\gamma X = \{r\gamma x \mid x \in X\}$. Then we have

(i) $r \circ \gamma \circ (X_1 \oplus X_2) = r \cdot \gamma \cdot (X_1 \oplus X_2)$

$= r \cdot \gamma \cdot ((X_1 \backslash X_2) \cup (X_2 \backslash X_1)) = r \cdot \gamma \cdot (\{a \mid a \in (X_1 \backslash X_2) \cup (X_2 \backslash X_1)\}$

$= \{r \cdot \gamma \cdot a \mid a \in (X_1 \backslash X_2) \cup (X_2 \backslash X_1)\}$.

And

$r \circ \gamma \circ X_1 \oplus r \circ \gamma \circ X_2 = r \cdot \gamma \cdot X_1 \oplus r \cdot \gamma \cdot X_2$

$= (r \cdot \gamma \cdot X_1 \backslash r \cdot \gamma \cdot X_2) \cup (r \cdot \gamma \cdot X_2 \backslash r \cdot \gamma \cdot X_1)$

$= \{r \cdot \gamma \cdot x \mid x \in (X_1 \backslash X_2)\} \cup \{r \cdot \gamma \cdot x \mid x \in (X_2 \backslash X_1)\}.$

$= \{r \cdot \gamma \cdot x \mid x \in (X_1 \backslash X_2) \cup (X_2 \backslash X_1)\}.$

(ii) $(r_1 + r_2) \circ \gamma \circ X = (r_1 + r_2) \cdot \gamma \cdot X$

$= \{(r_1 + r_2) \cdot \gamma \cdot x \mid x \in X\} = \{r_1 \cdot \gamma \cdot x + r_2 \cdot \gamma \cdot x \mid x \in X\}$

$= r_1 \cdot \gamma \cdot X + r_2 \cdot \gamma \cdot X = r_1 \circ \gamma \circ X + r_2 \circ \gamma \circ X.$

(iii) $r \circ (\gamma_1 + \gamma_2) \circ X = r \cdot (\gamma_1 + \gamma_2) \cdot X$

$= \{r \cdot (\gamma_1 + \gamma_2) \cdot x \mid x \in X\} = \{r \cdot \gamma_1 \cdot x + r \cdot \gamma_2 \cdot x \mid x \in X\}$

$= r \cdot \gamma_1 \cdot X + r \cdot \gamma_2 \cdot X = r \circ \gamma_1 \circ X + r \circ \gamma_2 \circ X.$

(iv) $r_1 \circ \gamma_1 \circ (r_2 \circ \gamma_2 \circ X)$

$= r_1 \cdot \gamma_1 \cdot (r_2 \circ \gamma_2 \circ X)$

$= \{r_1.\gamma_1.(r_2 \circ \gamma_2 \circ x) | x \in X\}$

$= \{r_1.\gamma_1.(r_2.\gamma_2.x) \mid x \in X\} = \{(r_1.\gamma_1.r_2).\gamma_2.x \mid x \in X\} = (r_1.\gamma_1.r_2).\gamma_2.X.$

Corollary 3.13. If in Proposition 3.12, we define $\oplus$ by $A \oplus B = \{a + b | a \in A, b \in B\}$. Then $(Sub(M), \oplus, \circ)$ is an R_Γ-module.

Proposition 3.14. Let $(R, \circ)$ and $(S, \bullet)$ be Γ-rings. Let $(M, .)$ be a left R_Γ-module and right S_Γ-module. Then $A = \{\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \mid r \in R, s \in S, m \in M\}$ is a Γ-ring and A_Γ-module under the mappings

$$\star : A \times \Gamma \times A \longrightarrow A$$
$$(\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}, \gamma, \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix}) \longmapsto \begin{pmatrix} r \circ \gamma \circ r_1 & r.\gamma.m_1 + m.\gamma.s_1 \\ 0 & s \bullet \gamma \bullet s_1 \end{pmatrix}.$$

□

Proof. Straightforward.

Example 3.15. Let $(R, \circ)$ be a Γ-ring . Then $R \oplus \mathbb{Z} = \{(r, s) \mid r \in R, s \in \mathbb{Z}\}$ is an left R_Γ-module, where $\oplus$ addition operation is defined $(r, n) \oplus (r', n') = (r +_R r', n +_{\mathbb{Z}} n')$ and the product $\cdot : R \times \Gamma \times (R \oplus \mathbb{Z}) \longrightarrow R \oplus \mathbb{Z}$ is defined $r' \cdot \gamma \cdot (r, n) \longrightarrow (r' \circ \gamma \circ r, n)$.

Example 3.16. Let R be the set of all digraphs (A digraph is a pair (V, E) consisting of a finite set V of vertices and a subset E of $V \times V$ of edges) and define addition on R by setting $(V_1, E_1) + (V_2, E_2) = (V_1 \cup V_2, E_1 \cup E_2)$. Obviously R is a commutative group since $(\emptyset, \emptyset)$ is the identity element and the inverse of every element is itself. For $\Gamma \subseteq R$ consider the mapping

$$\cdot : R \times \Gamma \times R \longrightarrow R$$
$$(V_1, E_1) \cdot (V_2, E_2) \cdot (V_3, E_3) = (V_1 \cup V_2 \cup V_3, E_1 \cup E_2 \cup E_3 \cup \{V_1 \times V_2 \times V_3\}),$$

under condition

$$\begin{aligned}(\emptyset, \emptyset) &= (\emptyset, \emptyset) \cdot (V_1, E_1) \cdot (V_2, E_2)(V_1, E_1) \cdot (\emptyset, \emptyset) \cdot (V_2, E_2)\\ &= (V_1, E_1) \cdot (\emptyset, \emptyset) \cdot (V_2, E_2)\\ &= (V_1, E_1) \cdot (V_2, E_2) \cdot (\emptyset, \emptyset).\end{aligned}$$

It is easy to verify that R is an R_Γ-module .

Example 3.17. Suppose that M is an abelian group. Set $R = M_{mn}$ and $\Gamma = M_{nm}$, so by definition of multiplication matrix subset $R^{(t)}_{mn} = \{(x_{ij}) \mid x_{tj} = 0 \ \forall \ j = 1, ...m\}$ is a right R_Γ-module. Also, $C^{(k)}_{mn} = \{x_{ij}) \mid x_{ik} = 0 \ \forall i = 1, ..., n\}$ is a left R_Γ-module.

Example 3.18. Let $(M, \bullet)$ be an R_Γ-module over Γ-ring $(R, .)$ and $S = \{(a, 0) | a \in R\}$. Then $R \times M = \{(a, m) | a \in R, m \in M\}$ is an S_Γ-module, where addition operation is defined by $(a, m) \oplus (b, m_1) = (a +_R b, m +_M m_1)$. Obviously, $(R \times M, \oplus)$ is an additive group. Now consider the mapping

$$\circ : S \times \Gamma \times (R \times M) \longrightarrow R \times M$$
$$((a, 0), \gamma, (b, m)) \longmapsto (a, 0) \circ \gamma \circ (b, m) = (a.\gamma.b, a \bullet \gamma \bullet m).$$

Then it is easy to verify that $R \times M$ is an S_Γ-module.

Example 3.19 Let R be a Γ-ring and $(M, .)$ be an R_Γ-module. Consider the mapping $\alpha : M \longrightarrow R$. Then M is an M_Γ-module, under the mapping

$$\circ : M \times \Gamma \times M \longrightarrow M$$

$$(m, \gamma, n) \longmapsto m \circ \gamma \circ n = (\alpha(m)).\gamma.n.$$

Example 3.20. Let $(R, \cdot)$ and $(S, \circ)$ be Γ- rings. Then

(i) The product $R \times S$ is a Γ- ring, under the mapping

$$((r_1, s_1), \gamma, (r_2, s_2)) \longmapsto (r_1 \cdot \gamma \cdot r_2, s_1 \circ \gamma \circ s_2).$$

(ii) For $A = \{ \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \mid r \in R, s \in S\}$ there exists a mapping $R \times S \longrightarrow A$, such that $(r, s) \longrightarrow \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$ and A is a Γ- ring. Moreover, A is an $(R \times S)_\Gamma$- module under the mapping

$$(R \times S) \times \Gamma \times A \longrightarrow A \ ((r_1, s_1), \gamma, \begin{pmatrix} r_2 & 0 \\ 0 & s_2 \end{pmatrix}) \longrightarrow \begin{pmatrix} r_1 \cdot \gamma \cdot r_2 & 0 \\ 0 & s_1 \circ \gamma \circ s_2 \end{pmatrix}.$$

Example 3.21. Let $(R, \cdot)$ be a Γ-ring. Then $R \times R$ is an R_Γ-module and $(R \times R)_\Gamma$- module. Consider addition operation $(a, b) + (c, d) = (a +_R c, b +_R d)$. Then $(R \times R, +)$ is an additive group. Now define the mapping $R \times \Gamma \times (R \times R) \longmapsto R \times R$ by $(r, \gamma, (a, b)) \longmapsto (r \cdot \gamma \cdot a, r \cdot \gamma \cdot b)$ and $(R \times R) \times \Gamma \times (R \times R) \longrightarrow R \times R$ by $((a, b), \gamma, (c, d)) \longmapsto (a \cdot \gamma \cdot c + b \cdot \gamma \cdot d, a \cdot \gamma \cdot d + b \cdot \gamma \cdot c)$. Then $R \times R$ is an $(R \times R)_\Gamma$- module.

4 Submodules of Gamma Modules

In this section we study submodules of gamma modules and investigate their properties. In the sequel R denotes a Γ-ring and all gamma modules are R_Γ-modules

Definition 4.1. Let $(M, +)$ be an R_Γ-module. A nonempty subset N of $(M, +)$ is said to be a (left) R_Γ-submodule of M if N is a subgroup of M and $R\Gamma N \subseteq N$, where

$R\Gamma N = \{r\gamma n | \gamma \in \Gamma, r \in R, n \in N\}$, that is for all $n, n' \in N$ and for all $\gamma \in \Gamma, r \in R$; $n - n' \in N$ and $r\gamma n \in N$. In this case we write $N \leq M$.

Remark 4.2. (i) Clearly $\{0\}$ and M are two trivial R_Γ-submodules of R_Γ-module M, which is called trivial R_Γ-submodules.

(ii) Consider R as R_Γ-module. Clearly, every ideal of Γ-ring R is submodule, of R as R_Γ-module.

Theorem 4.3. Let M be an R_Γ-module. If N is a subgroup of M, then the factor group M/N is an R_Γ-module under the mapping $. : R \times \Gamma \times M/N \longrightarrow M/N$ is defined $(r, \gamma, m + N) \longmapsto (r.\gamma.m) + N$.

Proof. Straight forward.

Theorem 4.4. Let N be an R_Γ-submodules of M. Then every R_Γ-submodule of M/N is of the form K/N, where K is an R_Γ-submodule of M containing N.

Proof. For all $x, y \in K, x + N, y + N \in K/N$; $(x + N) - (y + N) = (x - y) + N \in K/N$, we have $x - y \in K$, and $\forall r \in R\ \forall \gamma \in \Gamma, \forall x \in K$, we have

$$r\gamma(x + N) = r\gamma x + N \in K/N \Rightarrow r\gamma x \in K.$$

Then K is a R_Γ-submodule M . Conversely,it is easy to verify that $N \subseteq K \leq M$ then K/N is R_Γ-submodule of M/N. This complete the proof. $\square$

Proposition 4.5. Let M be an R_Γ-module and I be an ideal of R. Let X be a nonempty subset of M. Then

$I\Gamma X = \{\sum_{i=1}^{n} a_i\gamma_i x_i \mid a_i \in Ir_{\gamma i} \in \Gamma, x_i \in X, n \in \mathbb{N}\}$ is an R_Γ-submodule of M.

Proof. (i) For elements $x = \sum_{i=1}^{n} a_i\alpha_i x_i$ and $y = \sum_{j=1}^{m} x_{a'_j \beta_j y_j}$ of $I\Gamma X$, we have

$$x - y = \sum_{k=1}^{m+n} b_k\gamma_k z_k \in I\Gamma X.$$

Now we consider the following cases:

Case (1): If $1 \leq k \leq n$, then $b_k = a_k, \gamma_k = \alpha_k, z_k = x_k$.

Case(2): If $n+1 \leq k \leq m+n$, then $b_k = -a'_{k-n}, \gamma_k = \beta_{k-n}, z_k = y_{k-n}$. Also (ii) $\forall r \in R, \forall \gamma \in \Gamma, \forall a = \sum_{i=1}^{n} a_i\gamma_i x_i \in I\Gamma X$, we have $r\gamma x = \sum_{i=1}^{n} r\gamma(a_i\gamma_i x_i) = \sum_{i=1}^{n}(r\gamma a_i)\gamma_i x_i$. Thus $I\Gamma X$ is an R_Γ-submodule of M. $\square$

Corollary 4.6. If M is an R_Γ-module and S is a submodule of M. Then $R\Gamma S$ is an R_Γ-submodule of M.

Let $N \leq M$. Define $N : M = \{r \in R | r\gamma m \quad \forall \gamma \in \Gamma \ \ \forall m \in M\}$.

It is easy to see that $N : M$ is an ideal of Γ ring R.

Theorem 4.7. Let M be an R_Γ-module and I be an ideal of R. If $I \subseteq (0 : M)$, then M is an $(R/I)_\Gamma$-module.

proof. Since R/I is Γ-ring, $define the mapping \bullet : (R/I) \times \Gamma \times M \longrightarrow M$ by

$(r+I, \gamma, m) \longmapsto r\gamma m.$. The mapping $\bullet$ is well-defined since $I \subseteq (0 : M)$. Now it is straight forward to see that M is an $(R/I)_\Gamma$-module. $\square$

Proposition 4.8. Let R be a Γ-ring, I be an ideal of R, and $(M, .)$ be a R_Γ-module. Then $M/(I\Gamma M)$ is an $(R/I)_\Gamma$- module.

Proof. First note that $M/(I\Gamma M)$ is an additive subgroup of M. Consider the mapping

$$\gamma \bullet (m + I\Gamma M) = r.\gamma.m + I\Gamma M$$

$)Now it is straight forward to see that$ M$is an$ (R/I)$_\Gamma$-module. $\square$

Proposition 4.9. Let M be an R_Γ-module and $N \leq M$, $m \in M$. Then

$(N : m) = \{a \in R \mid a\gamma m \in N \ \forall \gamma \in \Gamma\}$ is a left ideal of R.

Proof. Obvious.

Proposition 4.10. If N and K are R_Γ-submodules of a R_Γ-module M and if A, B are nonempty subsets of M then:

(i) $A \subseteq B$ implies that $(N : B) \subseteq (N : A)$;

(ii) $(N \cap K : A) = (N : A) \cap (K : A)$;

(iii) $(N : A) \cap (N : B) \subseteq (N : A + B)$, moreover the equality hold if $0_M \in A \cap B$.

proof. (i) Easy.

(ii) By definition, if $r \in R$, then $r \in (N \cap K : A) \Longleftrightarrow \forall a \in Ar \in (N \cap K : a) \Longleftrightarrow \forall \gamma \in \Gamma;\ r\gamma a \in N \cap K \Longleftrightarrow r \in (N : A) \cap K : A)$.

(iii) If $r \in (N : A) \cap (N : B)$. Then $\forall \gamma \in \Gamma, \forall a \in A, \forall b \in B,\ r\gamma(a + b) \in N$ and $r \in (N : A + B)$.

Conversely, $0_M \in A + B \Longrightarrow A \cup B \subseteq A + B \Longrightarrow (N : A + B) \subseteq (N : A \cup B)$ by(i).

Again by using $A, B \subseteq A \cup B$ we have $(N : A \cup B) \subseteq (N : A) \cap (N : B)$. $\square$

Definition 4.11. Let M be an R_Γ-module and $\emptyset \neq X \subseteq M$. Then the generated R_Γ-submodule of M, denoted by $< X >$ is the smallest R_Γ-submodule of M containing X, i.e. $< X >= \cap\{N|N \leq M\}$, X is called the *generator* of $< X >$; and $< X >$ is finitely generated if $|X| < \infty$. If $X = \{x_1, ...x_n\}$ we write $< x_1, ..., x_n >$ instead $< \{x_1, ..., x_n\} >$. In particular, if $X = \{x\}$ then $< x >$ is called the *cyclic submodule* of M, generated by x.

Lemma 4.12. Suppose that M is an R_Γ-module. Then

(i) Let $\{M_i\}_{i\in I}$ be a family of R_Γ-submodules M. Then $\cap M_i$ is the largest R_Γ-submodule of M, such that contained in M_i, for all $i \in I$.

(ii) If X is a subset of M and $|X| < \infty$. Then

$< X >= \{\sum_{i=1}^{m} n_i x_i + \sum_{j=1}^{k} r_j\gamma_j x_j | k, m \in \mathbb{N}, n_i \in \mathbb{Z}, \gamma_j \in \Gamma, r_j \in R, x_i, x_j \in X\}$.

Proof. (i) It is easy to verify that $\cap_{i\in I} M_i \subseteq M_i$ is a R_Γ-submodule of M. Now suppose that $N \leq M$ and $\forall\ i \in I, N \subseteq M_i$, then $N \subseteq \cap M_i$.

(ii) Suppose that the right hand in (b) is equal to D. First, we show that D is an R_Γ-submodule containing X. $X \subseteq D$ and difference of two elements of D is belong to D and $\forall r \in R\ \forall \gamma \in \Gamma, \forall a \in D$ we have

$$r\gamma a = r\gamma(\sum_{i=1}^{m} n_i x_i + \sum_{j=1}^{k} r_j\gamma_j x_j) = \sum_{i=1}^{m} n_i(r\gamma x_i) + \sum_{j=1}^{k}(r\gamma r_j)\gamma_j x_j \in D.$$

Also, every submodule of M containing X, clearly contains D. Thus D is the smallest

R_Γ-submodules of M, containing X. Therefore $< X >= D$. $\square$

For $N, K \leq M$, set $N + K = \{n + k | n \in N, K \in K\}$. Then it is easy to see that $M + N$ is an R_Γ-submodules of M, containing both N and K. Then the next result immediately follows.

Lemma 4.13. Suppose that M is an R_Γ-module and $N, K \leq M$. Then $N + K$ is the smallest submodule of M containing N and K.

Set $L(M) = \{N | N \leq M\}$. Define the binary operations $\vee$ and $\wedge$ on $L(M)$ by $N \vee K = N + K$ and$N \wedge K = N \cap K$. In fact $(L(M), \vee, \wedge)$ is a lattice. Then the next result immediately follows from lemmas 4.12. 4.13.

Theorem 4.13. $L(M)$ is a complete lattice.

5 Homomorphisms Gamma Modules

In this section we study the homomorphisms of gamma modules. In particular we investigate the behavior of submodules od gamma modules under homomorphisms.

Definition 5.1. Let M and N be arbitrary R_Γ-modules. A mapping $f : M \longrightarrow N$ is a *homomorphism* of R_Γ-modules (or an R_Γ-homomorphisms) if for all $x, y \in M$ and $\forall r \in R, \forall \gamma \in \Gamma$ we have

(i) $f(x + y) = f(x) + f(y)$;

(ii) $f(r\gamma x) = r\gamma f(x)$.

A homomorphism f is *monomorphism* if f is one-to-one and f is *epimorphism* if f is onto. f is called *isomorphism* if f is both monomorphism and epimorphism. We denote the set of all R_Γ-homomorphisms from M into N by $Hom_{R_\Gamma}(M, N)$ or shortly by

$Hom_{R_\Gamma}(M,N)$. In particular if $M=N$ we denote $Hom(M,M)$ by $End(M)$.

Remark 5.2. If $f: M \longrightarrow N$ is an R_Γ-homomorphism, then $Kerf = \{x \in M | f(x) = 0\}$, $Imf = \{y \in N | \exists x \in M; y = f(x)\}$ are R_Γ-submodules of M.

Example 5.3. For all R_Γ-modules A, B, the zero map $0: A \longrightarrow B$ is an R_Γ-homomorphism.

Example 5.4. Let R be a Γ-ring. Fix $r_0 \in \Gamma$ and consider the mapping $\phi : R[x] \longrightarrow R[x]$ by $f \longmapsto f\gamma_0 x$. Then ϕ is an R_Γ-module homomorphism, because

$\forall r \in R,\ \forall \gamma \in \Gamma$ and $\ \forall f, g \in R[x]$:

$\phi(f+g) = (f+g)\gamma_0 x = f\gamma_0 x + g\gamma_0 x = \phi(f) + \phi(g)$ and

$\phi(r\gamma f) = r\gamma f\gamma_0 x = r\gamma\phi(f)$.

Example 5.5. If $N \leq M$, then the natural map $\pi : M \longrightarrow M/N$ with $\pi(x) = x + N$ is an R_Γ-module epimorphism with $ker\pi = N$.

Proposition 5.6. If M is unitary R_Γ-module and $End(M) = \{f : M \longrightarrow M | f \ \ is\ R_\Gamma - homomorphism\}$. Then M is an $End(M)_\Gamma$-module.

Proof. It is well known that $End(M)$ is an abelian group with usual addition of functions. Define the mapping

$$. : End(M) \times \Gamma \times M \longrightarrow M$$

$$(f, \gamma, m) \longmapsto f(1.\gamma.m) = 1\gamma f(m),$$

where 1 is the identity map. Now it is routine to verify that M is an $End(M)_\Gamma$-module.$\square$

Lemma 5.7. Let $f : M \longrightarrow N$ be an R_Γ-homomorphism. If $M_1 \leq M$ and $N_1 \leq$. Then

$(i)\ \ Kerf \leq M\ ,\ Imf \leq N;$

$(ii)\ \ f(M_1) \leq Imf;$

$(iii)\ \ Kerf^{-1}(N_1) \leq M.$

Example 5.8. Consider $L(M)$ the lattice of R_Γ-submodules of M. We know that $(L(M), +)$ is a monoid with the sum of submodules. Then $L(M)$ is R_Γ-semimodule under the mapping

$.: R \times \Gamma \times T \longrightarrow T$, such that $(r, \gamma, N) \longmapsto r.\gamma.N = r\gamma N = \{r\gamma n | n \in N\}$.

Example 5.9. Let $\theta : R \longrightarrow S$ be a homomorphism of Γ-rings and M be an S_Γ-module. Then M is an R_Γ-module under the mapping $\bullet : R \times \Gamma \times M \longrightarrow M$ by $(r, \gamma, m) \longmapsto r \bullet \gamma \bullet m = \theta(r)$. Moreover if M is an S_Γ-module then M is a R_Γ-module for $R \subseteq S$.

Example 5.10. Let $(M, .)$ be an R_Γ-module and $A \subseteq M$. Letting $M^A = \{f | f : A \longrightarrow M \ \ \textit{is a map}\}$. Then M^A is an R_Γ-module under the mapping $\circ : R \times \Gamma \times M^A \longrightarrow M^A$ defined by $(r, \gamma, f) \longmapsto r \circ \gamma \circ f = r\gamma f(a)$, since M^A is an additive group with usual addition of maps.

Example 5.11. Let$(M, .)$ and $(N, \bullet)$ be R_Γ-modules. Then $Hom(M, N)$ is a R_Γ-module, under the mapping

$$\circ : R \times \Gamma \times Hom(M, N) \longrightarrow Hom(M, N)$$
$$(r, \gamma, \alpha) \longmapsto r \circ \gamma \circ \alpha,$$
$$\text{where } (r \bullet \gamma \bullet \alpha)(m) = r\gamma\alpha)(m).$$

Example 5.12. Let M be a left R_Γ-module and right S_Γ-module. If N be an R_Γ-module, then

(i) $Hom(M, N)$ is a left S_Γ-module. Indeed

$$\circ : S \times \Gamma \times Hom(M, N) \longrightarrow Hom(M, N)$$
$$(s, \gamma, \alpha) \longrightarrow s \circ \gamma \circ \alpha : M \longrightarrow N$$
$$m \longmapsto \alpha(m\gamma s)$$

(ii) $Hom(N, M)$ is right S_Γ-module under the mapping

$$\circ : Hom(N, M) \times \Gamma \times S \longrightarrow Hom(N, M)$$

$$(\alpha, \gamma, s) \longmapsto \alpha \circ \gamma \circ s : N \longrightarrow M$$

$$n \longmapsto \alpha(n).\gamma.s$$

Example 5.13. Let M be a left R_Γ-module and right S_Γ-module and $\alpha \in End(M)$ then α induces a right $S[t]_\Gamma$-module structure on M with the mapping

$$\circ : M \times \Gamma \times S[t] \longrightarrow M$$

$$(m, \gamma, \sum_{i=0}^n s_i t^i) \longmapsto m \circ \gamma \circ (\sum_{i=0}^n s_i t^i) = \sum_{i=0}^n (m\gamma s_i)\alpha^i$$

Proposition 5.14. Let M be a R_Γ-module and $S \subseteq M$. Then $S\Gamma M = \{\sum s_i \gamma_i a_i \mid s_i \in S, a_i \in M, \gamma_i \in \Gamma\}$ is an R_Γ-submodule of M.

Proof. Consider the mapping

$$\circ : R \times \Gamma \times (S\Gamma M) \longrightarrow S\Gamma M$$

$$(r, \gamma, \sum_{i=1}^n s_i \gamma_i a_i) \longmapsto \sum_{i=1}^n s_i \gamma_i (r\gamma a_i).$$

Now it is easy to check that $S\Gamma M$ is a R_Γ-submodule of M.

Example 5.16. Let $(R, .)$ be a Γ-ring. Let $\mathbb{Z}_2$, the cyclic group of order 2. For a nonempty subset A, set $Hom(R, \mathbb{B}^A) = \{f : R \longrightarrow \mathbb{B}^A\}$. Clearly $(Hom(R, \mathbb{B}^A), +)$ is an abelian group. Consider the mapping

$\circ : R \times \Gamma \times Hom(R, \mathbb{B}^A) \longrightarrow Hom(R, \mathbb{B}^A)$ that is defined

$$(r, \gamma, f) \longmapsto r \circ \gamma \circ f,$$

where $(r \circ \gamma \circ f)(s) : A \longrightarrow \mathbb{B}$ is defied by $(r \circ \gamma \circ f(s))(a) = f(s\gamma r)(a)$.

Now it is easy to check that $Hom(R, \mathbb{B}^A)$ is an Γ-ring.

Example 5.17. Let R and S be Γ-rings and $\varphi : R \longrightarrow S$ be a Γ-rings homomorphism. Then every S_Γ-module M can be made into an R_Γ-module by defining

$r\gamma x \ \ (r \in R, \gamma \in \Gamma, x \in M)$ to be $\varphi(r)\gamma x$. We says that the R_Γ-module structure M is given by pullback along φ.

Example 5.18. Let $\varphi : R \longrightarrow S$ be a homomorphism of Γ-rings then $(S, .)$ is an R_Γ-module. Indeed

$$\circ : R \times \Gamma \times S \longrightarrow S$$
$$(r, \gamma, s) \longmapsto r \circ \gamma \circ s = \varphi(r).\gamma.s$$

Example 5.19. Let $(M, +)$ be an R_Γ-module. Define the operation $\circ$ on M by $a \oplus b = b.a$. Then $(M, \oplus)$ is an R_Γ-module.

Proposition 5.20. Let R be a Γ-ring. If $f : M \longrightarrow N$ is an R_Γ-homomorphism and $C \leq kerf$, then there exists an unique R_Γ-homomorphism $\bar{f} : M/C \longrightarrow N$, such that for every $x \in M$; $Ker\bar{f} = Kerf/C \ and \ Im\bar{f} = Imf \ and \ \bar{f}(x + C) = f(x)$, also $\bar{f}$ is an R_Γ-isomorphism if and only if f is an R_Γ-epimorphism and $C = Kerf$. In particular $M/Kerf \cong Imf$.

Proof. Let $b \in x + C$ then $b = x + c$ for some $c \in C$, also $f(b) = f(x + c)$. We know f is R_Γ-homomorphism therefore $f(b) = f(x + c) = f(x) + f(c) = f(x) + 0 = f(x)$ (since $C \leq kerf$) then $\bar{f} : M/C \longrightarrow N$ is well defined function. Also $\forall \ x + C, y + C \in M/C$ and $\forall \ r \in R, \gamma \in \Gamma$ we have

(i) $\bar{f}((x+C)+(y+C)) = \bar{f}((x+y)+C) = f(x+y) = f(x)+f(y) = \bar{f}(x+C)+\bar{f}(y+C)$.

(ii) $\bar{f}(r\gamma(x + C)) = \bar{f}(r\gamma x + C) = f(r\gamma x) = r\gamma f(x) = r\gamma\bar{f}(x + C)$.

then $\bar{f}$ is a homomorphism of R_Γ-modules, also it is clear $Im\bar{f} = Imf$ and $\forall(x + C) \in ker\bar{f};\ x + C \in ker\bar{f} \Leftrightarrow \bar{f}(x + C) = 0 \Leftrightarrow f(x) = 0 \Leftrightarrow x \in kerf$ then $ker\bar{f} = kerf/C$.

Then definition $\bar{f}$ depends only f, then $\bar{f}$ is unique. $\bar{f}$ is epimorphism if and only if f is epimorphism. $\bar{f}$ is monomorphism if and only if $ker\bar{f}$ be trivial R_Γ-submodule of M/C. In actually if and only if $Kerf = C$ then $M/Kerf \cong Imf$.$\square$

Corollary 5.21. If R is a Γ-ring and M_1 is an R_Γ-submodule of M and N_1 is R_Γ-submodule of N, $f : M \longrightarrow N$ is a R_Γ-homomorphism such that $f(M_1) \subseteq N_1$ then f make a R_Γ-homomorphism $\bar{f} : M/M_1 \longrightarrow N/N_1$ with operation $m + M_1 \longmapsto f(m) + N_1$. $\bar{f}$ is R_Γ-isomorphism if and only if $Imf + N_1 = N,\ f^{-1}(N_1) \subseteq M_1$. In particular, if f is epimorphism such that $f(M_1) = N_1,\ kerf \subseteq M_1$ then f is a R_Γ-isomorphism.

proof. We consider the mapping $M \longrightarrow^f N \longrightarrow^\pi N/N_1$. In this case; $M_1 \subseteq f^{-1}(N_1) = ker\pi f\ (\forall m_1 \in M_1,\ f(m_1) \in N_1 \Rightarrow \pi f(m_1) = 0 \Rightarrow m_1 \in ker\pi f)$. Now we use Proposition 5.20 for map $\pi f : M \longrightarrow N/N_1$ with function $m \longmapsto f(m) + N_1$ and submodule M_1 of M.

Therefore, map $\bar{f} : M/M_1 \longrightarrow N/N_1$ that is defined $m + M \longmapsto f(m) + N_1$ is a R_Γ-homomorphism. It is isomorphism if and only if πf is epimorphism, $M_1 = ker\pi f$. But condition will satisfy if and only if $Imf + N_1 = N$, $f^{-1}(N_1) \subseteq M_1$. If f is epimorphism then $N = Imf = Imf + N_1$ and if $f(M_1) = N_1$ and $kerf \subseteq M_1$ then $f^{-1}(N_1) \subseteq M_1$ so $\bar{f}$ is isomorphism.$\square$

Proposition 5.22. Let B, C be R_Γ-submodules of M.

(i) There exists a R_Γ-isomorphism $B/(B \cap C) \cong (B + C)/C$.

(ii) If $C \subseteq B$, then B/C is an R_Γ-submodule of M/C and there is an R_Γ-isomorphism $(M/C)/(B/C) \cong M/B$.

Proof. (i) Combination $B \longrightarrow^j B + C \longrightarrow^\pi (B + C)/C$ is an R_Γ-homomorphism with kernel$= B \cap C$, because $ker\pi j = \{b \in B | \pi j(b) = 0_{(B+C)/C}\} = \{b \in B | \pi(b) = C\} = \{b \in B | b + C = C\} = \{b \in B | b \in C\} = B \cap C$ therefore, in order to Proposition 5.20., $B/(B \cap C) \cong Im(\pi j)(\star)$, every element of $(B + C)/C$ is to form $(b + c) + C$, thus $(b + c) + C = b + C = \pi j(b)$ then πj is epimorphism and $Im\pi j = (B + C)/C$ in attention $(\star)$, $B/(B \cap C) \cong (B + C)/C$.

(ii) We consider the identity map $i : M \longrightarrow M$, we have $i(C) \subseteq B$, then in order to apply Proposition 5.21. we have R_Γ-epimorphism $\bar{i} : M/C \longrightarrow M/B$ with $\bar{i}(m + C) = m + B$ by using (i). But we know $B = \bar{i}(m + C)$ if and only if $m \in B$ thus $ker\ \bar{i} = \{m + C \in M/C | m \in B\} = B/C$ then $ker\bar{i} = B/C \leq M/C$ and we have

$$M/B = Im\bar{i} \cong (M/C)/(B/C).\square$$

Let M be a R_Γ-module and $\{N_i | i \in \Omega\}$ be a family of R_Γ-submodule of M. Then $\cap_{i\in\Omega} N_i$ is a R_Γ-submodule of M which, indeed, is the largest R_Γ-submodule M contained in each of the N_i. In particular, if A is a subset of a left R_Γ-module$M$ then intersection of all submodules of M containing A is a R_Γ-submodule of M, called the submodule *generated* by A. If A generates all of the R_Γ-module, then A is a set of*generators* for M. A left R_Γ-module having a finite set of generators is *finitely generated*. An element m of the R_Γ-submodule generated by a subset A of a R_Γ-module M is a *linear combination* of the elements of A.

If M is a left R_Γ-module then the set $\sum_{i\in\Omega} N_i$ of all finite sums of elements of N_i is an R_Γ-submodule of M generated by $\cup_{i\in\Omega} N_i$. R_Γ-submodule generated by $X = \cup_{i\in\Omega} N_i$ is $D = \{\sum_{i=1}^{s} r_i\gamma_i a_i + \sum_{j=1}^{t} n_j b_j | a_i, b_j \in X, r_i \in R, n_j \in \mathbb{Z}, \gamma_i \in \Gamma\}$ if M is a unitary R_Γ-module then $D = R\Gamma X = \{\sum_{i=1}^{s} r_i\gamma_i a_i | r_i \in R, \gamma_i \in \Gamma, a_i \in X\}$.

Example 5.23. Let M, N be R_Γ-modules and $f, g : M \longrightarrow N$ be R_Γ-module homomorphisms. Then $K = \{m \in M \mid f(m) = g(m)\}$ is R_Γ-submodule of M.

Example 5.24. Let M be a R_Γ-module and let N, N' be R_Γ- submodules of M. Set $A = \{m \in M \mid m + n \in N'\ for\ some\ n \in N\}$ is an R_Γ-module of M containing N'.

Proposition 5.25. Let $(M, \cdot)$ be an R_Γ- module and M generated by A. Then there exists an R_Γ-homomorphism $R^{(A)} \longrightarrow M$, such that $f \longmapsto \sum_{a\in A, a\in supp(f)} f(a) \cdot \gamma \cdot a$.

Remark 5.26. Let R be a Γ- ring and let $\{(M_i, o_i) | i \in \Omega\}$ be a family of left R_Γ-modules. Then $\times_{i\in\Omega} M_i$, the Cartesian product of M_i's also has the structure of a left R_Γ-module under componentwise addition and mapping

$$\cdot : R \times \Gamma \times (\times M_i) \longrightarrow \times M_i$$
$$(r, \gamma, \{m_i\}) \longrightarrow r \cdot \gamma \cdot \{m_i\} = \{r o_i \gamma o_i m_i\}_\Omega.$$

We denote this left R_Γ-module by $\prod_{i\in\Omega} M_i$. Similarly, $\sum_{i\in\Omega} M_i = \{\{m_i\} \in \prod M_i | m_i = 0 \quad \text{for all but finitely many indices} \quad i\}$ is a R_Γ-submodule of $\prod_{i\in\Omega} M_i$. For each h in Ω we have canonical R_Γ- homomorphisms $\pi_h : \prod M_i \longrightarrow M_h$ and $\lambda_h : M_h \longrightarrow \sum M_i$ is defined respectively by $\pi_h :< m_i > \longmapsto m_h$ and $\lambda(m_h) =< u_i >$, where

$$u_i = \begin{cases} 0 & i \neq h \\ m_h & i = h \end{cases}$$

The R_Γ-module $\prod M_i$ is called the (external)*direct product* of the R_Γ- modules M_i and the R_Γ- module $\sum M_i$ is called the (external) *direct sum* of M_i. It is easy to verify that if M is a left R_Γ-module and if $\{M_i | i \in \Omega\}$ is a family of left R_Γ-modules such that, for each $i \in \Omega$, we are given an R_Γ-homomorphism $\alpha_i : M \longrightarrow M_i$ then there exists a unique R_Γ- homomorphism $\alpha : M \longrightarrow \prod_{i\in\Omega} M_i$ such that $\alpha_i = \alpha\pi_i$ for each $i \in \Omega$. Similarly, if we are given an R_Γ-homomorphism $\beta_i : M_i \longrightarrow M$ for each $i \in \Omega$ then there exists an unique R_Γ- homomorphism $\beta : \sum_{i\in\Omega} M_i \longrightarrow M$ such that $\beta_i = \lambda_i\beta$ for each $i \in \Omega$.

Remark 5.27. Let M be a left R_Γ-module. Then M is a right R_Γ^{op}-module under the mapping

$$* : M \times \Gamma \times R^{op} \longrightarrow M$$
$$(m, \gamma, r) \longmapsto m * \gamma * r = r\gamma m.$$

Definition 5.28. A nonempty subset N of a left R_Γ-module M is *subtractive* if and only if $m + m' \in N$ and $m \in N$ imply that $m' \in N$ for all $m, m' \in M$. Similarly, N is *strong subtractive* if and only if $m + m' \in N$ implies that $m, m' \in N$ for all $m, m' \in M$.

Remark 5.29. (i) Clearly, every submodule of a left R_Γ-module is subtractive. Indeed, if N is a R_Γ-submodule of a R_Γ-module M and $m \in M, n \in N$ are elements satisfying $m + n \in N$ then $m = (m + n) + (-n) \in N$.

(ii) If $N, N' \subseteq N$ are R_Γ-submodules of an R_Γ-module M, such that N' is a subtractive R_Γ-submodule of N and N is a subtractive R_Γ-submodule of M then N' is a subtractive R_Γ-module of M.

Note. If $\{M_i | i \in \Omega\}$ is a family of (resp. strong) subtractive R_Γ-submodule of a left R_Γ-module M then $\cap_{i\in\Omega} M_i$ is again (resp. strong) subtractive. Thus every R_Γ -submodule of a left R_Γ-module M is contained in a smallest (resp. strong) subtractive R_Γ-submodule of M, called its (resp. strong) *subtractive closure* in M.

Proposition 5.30 Let R be a Γ-ring and let M be a left R_Γ -module. If N, N' and $N'' \leq M$ are submodules of M satisfying the conditions that N is subtractive and $N' \subseteq N$, then $N \cap (N' + N'') = N' + (N \cap N'')$.

Proof. Let $x \in N \cap (N' + N'')$. Then we can write $x = y + z$, where $y \in N'$ and $z \in N''$. by $N' \subseteq N$, we have $y \in N$ and so, $z \in N$, since N is subtractive. Thus $x \in N' + (N \cap N'')$, proving that $N \cap (N' + N'') \subseteq N' + (N \cap N'')$. The reverse containment is immediate.□

Proposition 5.31. If N is a subtractive R_Γ-submodule of a left R_Γ-module M and if A is a nonempty subset of M then $(N : A)$ is a subtractive left ideal of R.

Proof. Since the intersection of an arbitrary family of subtractive left ideals of R is again subtractive, it suffices to show that $(N : m)$ is subtractive for each element m. Let $a \in R$ and $b \in (N : M)$ (for $\gamma \in \Gamma$) satisfy the condition that $a + b \in (N : M)$. Then $a\gamma m + b\gamma m \in N$ and $b\gamma m \in N$ so $a\gamma m \in N$, since N is subtractive. Thus $a \in (N : M)$.□.

proposition 5.32. If I is an ideal of a Γ-ring R and M is a left R_Γ-module. Then

$N = \{m \in M \mid I\Gamma m = \{0\}\}$ is a subtractive R_Γ-submodule of M.

Proof. Clearly, N is an R_Γ-submodule of M. If $m, m' \in M$ satisfy the condition that m and $m + m'$ belong to N then for each $r \in I$ and for each $\gamma \in \Gamma$ we have $0 = r\gamma(m + m') = r\gamma m + r\gamma m'm' = r\gamma m'$, and hence $m' \in N$. Thus N is subtractive. $\square$

proposition 5.33. Let $(R, +, \cdot)$ be a Γ-ring and let M be an R_Γ-module and there exists bijection function $\delta : M \longrightarrow R$. Then M is a Γ-ring and M_Γ-module.

Proof. Define $\circ : M \times \Gamma \times M \longrightarrow M$ by $(x, \gamma, y) \longmapsto x \circ \gamma \circ y = \delta^{-1}(\delta(x) \cdot \gamma\delta(y))$. It is easy to verify that R is a Γ- ring. If M is a set together with a bijection function $\delta : X \longrightarrow R$ then the Γ-ring structure on R induces a Γ-ring structure $(M, \oplus, \odot)$ on X with the operations defined by $x \oplus y = \delta^{-1}(\delta(x) + \delta(y))$ and

$$x \odot \gamma \odot y = \delta^{-1}(\delta(x) \cdot \gamma \cdot \delta(y)).\square$$

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