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# A Common Fixed Point Theorem For Three Weakly Compatible Selfmaps Of A S-metric Space

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**Abstract**

Fixed point theorems were established by using contractive conditions. In this paper we prove a common fixed point theorem for three weakly compatible selfmaps of a S-metric space by utilizing a contractive condition of rational type.Further we deduce a common fixed point theorem for two weakly compatible selfmaps of a S-metric space.
**Keywords**: S-metric space; Fixed point; Weakly compatible mappings; Associated sequence of a point relative to three selfmaps.
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# 1 Introduction

Fixed point theory is an important branch of non-linear analysis due to its application potential. In proving fixed point theorems, we use completeness, continuity, convergence and various other topological aspects. Banach's Contraction Principle or Banach's fixed point theorem is one of the most important results in nonlinear analysis. This theorem has been generalized in many directions by generalizing the underlying space or by viewing it as a common fixed point theorem along with other selfmaps.
In the past few years, a number of generalizations of metric spaces like G -metric spaces, partial metric spaces and cone metric spaces were initiated. These generalizations were used to extend the scope of the study of fixed point theory. Recently one more generalization, namely S -metric spaces, was introduced by Sedghi S , Shobe N, Aliouche.A [2012]. Among all generalizations, S-metric spaces evinced a lot of interest in many researchers as they unified, extended, generalized and refined several existing results onto these S -metric spaces.
Commutativity plays an important role in proving common fixed point theorems. As it is a stronger requirement, Sessa [1982] introduced the notion of weakly commuting maps as a generalization of commuting maps. Afterwards the idea of compatibility was introduced by G. Jungck [1986]. Later on Jungck and Rhoades [1998]introduced the notion of weakly compatibility as a generalization of compatibility. They also proved that compatible mappings are weakly compatible but not conversely.
In this paper we establish a common fixed point theorem for three weakly compatible selfmaps of a S-metric space using a contractive condition of rational type. Our theorem which is established in the framework of S-metric spaces generalizes the theorem of Sumit Chandok [2018] which is proved in metric space.
Now we recall some basic definitions required in the sequel in section 2 and establish main results in section 3.

# 2 Preliminaries

We now recollect the essential definitions which are useful for our discussion.

**Definition 2.1.** *Let $Y$ be a nonempty set. A function*
$S : Y^3 \to [0, \infty)$ *is said to be* $S - metric$ *if it satisfies the following conditions for each* $\beta_1, \beta_2, \beta_3, \beta_4 \in Y$
$(i)$ $S(\beta_1, \beta_2, \beta_3) \geq 0$,
$(ii)$ $S(\beta_1, \beta_2, \beta_3) = 0$ *if and only if* $\beta_1 = \beta_2 = \beta_3$,
$(iii)$ $S(\beta_1, \beta_2, \beta_3) \leq S(\beta_1, \beta_1, \beta_4) + S(\beta_2, \beta_2, \beta_4) + S(\beta_3, \beta_3, \beta_4)$.
*Then* $(Y, S)$ *is said to be a S-metric space.*

**Example 2.1.** *Let* $Y = \mathbb{R}$ *and* $S : \mathbb{R}^3 \to [0, \infty)$ *be defined by* $S(\beta_1, \beta_2, \beta_3) = |\beta_2 + \beta_3 - 2\beta_1| + |\beta_2 - \beta_3|$ *for* $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$*, then* $(Y, S)$ *is a S-metric space.*

**Remark 2.1.** *It is shown in a S-metric space that* $S(\beta_1, \beta_1, \beta_2) = S(\beta_2, \beta_2, \beta_1)$ *for all* $\beta_1, \beta_2 \in Y$*.*

**Definition 2.2.** *Let* $(Y, S)$ *be an S-metric space. A sequence* $\{t_n\}$ *in Y said to convergent, if there is a* $t \in Y$ *such that* $S(t_n, t_n, t) \to 0$*; that is for each* $\epsilon > 0$*, there exists an* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$*, we have* $S(t_n, t_n, t) < \epsilon$ *and we denote this by* $\lim\limits_{n\to\infty} t_n = t$*.*

**Definition 2.3.** *Suppose* $(Y, S)$ *is an S-metric space. A sequence* $\{t_n\}$ *in Y is called a Cauchy sequence if to each* $\epsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that* $S(t_n, t_n, t) < \epsilon$ *for each* $n, m \geq n_0$ *.*

**Definition 2.4.** *Let* $(Y, S)$ *be an S-metric space, If there exists sequences* $\{t_n\}$ *and* $\{u_n\}$ *such that* $\lim\limits_{n\to\infty} t_n = t$ *and* $\lim\limits_{n\to\infty} u_n = u$ *then* $\lim\limits_{n\to\infty} S(t_n, t_n, u_n) = S(t, t, u)$*, then we say that* $S(t, u, v)$ *is continuous in t and u .*

**Definition 2.5.** *Suppose* $\phi$ *and* $\psi$ *self maps of a S-metric space* $(Y, S)$ *such that for every sequence* $\{t_n\}$ *in Y with* $\lim\limits_{n\to\infty} \psi t_n = \lim\limits_{n\to\infty} \phi t_n = t$ *for some* $t \in X$ *we have* $\lim\limits_{n\to\infty} S(\psi\phi t_n, \psi\phi t_n, \phi\psi t_n) = 0$*, then* $\psi$ *and* $\phi$ *are called compatible mappings.*

**Definition 2.6.** *In a S-metric space* $(Y, S)$*,two selfmaps* $\phi$ *and* $\psi$ *of Y are said to be weakly compatible if* $\phi\psi t = \psi\phi t$ *whenever* $\phi t = \psi t$ *for* $t \in Y$ *.*

**Definition 2.7.** *If* $\psi$*,* $\mu$ *and* $\phi$ *are self maps of a non empty set Y such that* $\psi(Y) \subseteq \phi(Y)$*, and* $\mu(Y) \subseteq \phi(Y)$ *then for any* $t_0 \in Y$*, if* $\{t_n\}$ *is a sequence in Y such that* $\phi t_{2n+1} = \psi t_{2n}$ *and* $\phi t_{2n+2} = \mu t_{2n+1}$ *for* $n \geq 1$ *then* $\{t_n\}$ *is called an associated sequence of* $t_0$ *relative to three selfmaps* $\psi$*,* $\mu$ *and* $\phi$*.*

# 3 Main Theorem

We now state our main theorem of the section.

**Theorem 3.1.** *Let P be a subset of a S-metric space* $(Y, S)$*,* $\psi$*,* $\mu$ *and* $\phi$ *are three selfmaps of P such that*

*(i)* $\psi(Y) \cup \mu(Y) \subseteq \phi(Y)$ *and* $(\phi(P), S)$ *is complete.*

*(ii)*

$$
\begin{aligned}
&S(\mu y_1, \mu y_1, \psi y_2) \\
&\leq k_1\{\frac{S(\phi y_1, \phi y_1, \mu y_1).S(\phi y_2, \phi y_2, \psi y_2)}{S(\phi y_1, \phi y_1, \phi y_2) + S(\phi y_1, \phi y_1, \psi y_2) + S(\phi y_2, \psi y_2, \mu y_1)}\} \\
&\qquad + k_2 S(\phi y_1, \phi y_1, \phi y_2)
\end{aligned}
$$

*for every $y_1, y_2 \in P$ and $k_1, k_2 \in [0, 1)$ with $2k_1 + k_2 < 1$.*

*(iii) The pairs $(\phi, \psi)$ and $(\phi, \mu)$ are weakly compatible.*
*Then $\psi, \mu$ and $\phi$ have a unique common fixed point.*

*Proof.* Let $t_0$ be a point in Y. Since $\psi(Y) \cup \mu(Y) \subseteq \phi(Y)$, we obtain an associated sequence $\{t_n\}$ in $Y$ such that

$$\phi t_{2n+1} = \mu t_{2n}, \phi t_{2n+2} = \psi t_{2n+1}.$$

From the condition (ii) of Theorem 3.1 we have,

$$
\begin{aligned}
&S(\phi t_{2n+1}, \phi t_{2n+1}, \phi t_{2n+2}) \\
&= S(\mu t_{2n}, \mu t_{2n}, \psi t_{2n+1}) \\
&\leq k_1 \left[\frac{S(\phi t_{2n}, \phi t_{2n}, \mu t_{2n}).S(\phi t_{2n+1}, \phi t_{2n+1}, \psi t_{2n+1})}{S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) + S(\phi t_{2n}, \phi t_{2n}, \psi t_{2n+1}) + S(\phi t_{2n+1}, \phi t_{2n+1}, \mu t_{2n})}\right] \\
&\qquad + k_2 S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) \\
&\leq k_1 \left[\frac{S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}).S(\phi t_{2n+1}, \phi t_{2n+1}, \phi t_{2n+2})}{S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) + S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+2}) + S(\phi t_{2n+1}, \phi t_{2n+1}, \phi t_{2n+1})}\right] \\
&\qquad + k_2 S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) \\
&\leq 2k_1 \left[\frac{S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}).S(\phi t_{2n+1}, \phi t_{2n+1}, \phi t_{2n+2})}{S(\phi t_{2n+1}, \phi t_{2n+1}, \phi t_{2n+1})}\right] + k_2 S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) \\
&\leq (2k_1 + k_2) \quad S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}).
\end{aligned}
$$

Similarly, we can prove

$$S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}) \leq (2k_1 + k_2) S(\phi t_{2n-1}, \phi t_{2n-1}, \phi t_{2n})$$

Therefore

$$
\begin{aligned}
S(\phi t_n, \phi t_n, \phi t_{n+1}) &\leq (2k_1 + k_2) \quad S(\phi t_{n-1}, \phi t_{n-1}, \phi t_n) \\
&\leq (2k_1 + k_2)^2 \quad S(\phi t_{n-2}, \phi t_{n-2}, \phi t_{n-1}) \\
&\leq (2k_1 + k_2)^3 \quad S(\phi t_{n-3}, \phi t_{n-3}, \phi t_{n-2}) \\
&\cdots \qquad \cdots \qquad \cdots \\
&\cdots \qquad \cdots \qquad \cdots \\
&\leq (2k_1 + k_2)^n \quad S(\phi t_0, \phi t_0, \phi t_1) \to 0,
\end{aligned}
$$

since $(2k_1 + k_2)^n \to 0$ as $n \to \infty$.
Now we claim that $\{\phi t_n\}$ is a Cauchy sequence.
For any $m, n \in \mathbb{N}$ such that $m > n$ we have,

$$\begin{aligned} S(\phi t_n, \phi t_n, \phi t_m) &\leq 2[S(\phi t_n, \phi t_n, \phi t_{n+1}) + S(\phi t_{n+1}, \phi t_{n+1}, \phi t_{n+2}) + \cdots \\ &\qquad + S(\phi t_{m-1}, \phi t_{m-1}, \phi t_m)] \\ &\leq 2c^n S(\phi t_0, \phi t_0, \phi t_1) + c^{n+1} S(\phi t_0, \phi t_0, \phi t_1) + \cdots \\ &\qquad + c^m S(\phi t_0, \phi t_0, \phi t_1) \\ &\leq 2c^n (1 + c + c^2 + \cdots + c^{m-n}) S(\phi t_0, \phi t_0, \phi t_1) \\ &\leq 2c^n \frac{1 - c^{m-n}}{1 - c} S(\phi t_0, \phi t_0, \phi t_1) \\ &\leq 2 \frac{c^n}{1 - c} S(\phi t_0, \phi t_0, \phi t_1) \to 0, \end{aligned}$$

since $c < 1$ then $c^n \to 0$ as $n \to \infty$.
Therefore $\{\phi t_n\}$ is a Cauchy sequence in $X$.
Since $(\phi(P), S)$ is complete, there is a $t \in P$ such that $\phi t_n \to \phi t$ as $n \to \infty$.
We now prove that $t$ is a point of coincidence of $\mu, \psi$ and $\phi$.
From the condition (ii) of Theorem 3.1 we have,

$$\begin{aligned} &S(\phi t_{2n+1}, \phi t_{2n+1}, \psi t) \\ &= S(\mu t_{2n}, \mu t_{2n}, \psi t) \\ &\leq k_1 \left[ \frac{S(\phi t_{2n}, \phi t_{2n}, \mu t_{2n}).S(\phi t, \phi t, \psi t)}{S(\phi t_{2n}, \phi t_{2n}, \phi t) + S(\phi t_{2n}, \phi t_{2n}, \psi t) + S(\phi t, \phi t, \mu t_{2n})} \right] \\ &\qquad + k_2 S(\phi t_{2n}, \phi t_{2n}, \phi t) \\ &\leq k_1 \left[ \frac{S(\phi t_{2n}, \phi t_{2n}, \phi t_{2n+1}).S(\phi t, \phi t, \psi t)}{S(\phi t_{2n}, \phi t_{2n}, \phi t) + S(\phi t_{2n}, \phi t_{2n}, \psi t) + S(\phi t, \phi t, \phi t_{2n+1})} \right] \\ &\qquad + k_2 S(\phi t_{2n}, \phi t_{2n}, \phi t), \end{aligned}$$

which gives $S(\phi t, \phi t, \psi t) = 0$ as $n \to \infty$ and hence $\phi t = \psi t$.
Also we have,

$$\begin{aligned} &S(\mu t, \mu t, \phi t) \\ &= S(\mu t, \mu t, \psi t) \\ &\leq k_1 \left[ \frac{S(\phi t, \phi t, \mu t).S(\phi t, \phi t, \psi t)}{S(\phi t, \phi t, \phi t) + S(\phi t, \phi t, \psi t) + S(\phi t, \phi t, \mu t)} \right] + k_2 S(\phi t, \phi t, \phi t), \end{aligned}$$

which implies $S(\mu t, \mu t, \phi t) = 0$ proving $\mu t = \phi t$.
Therefore we have $\phi t = \psi t = \mu t = a(say)$,

proving that $t$ is a coincident point of $\mu, \psi$ and $\phi$.
Since the pairs $(\phi, \mu)$ and $(\phi, \psi)$ are weakly compatible, we have $\phi\mu t = \mu\phi t$ and $\phi\psi t = \psi\phi t$ which implies $\phi a = \psi a = \mu a$.
Now we have,

$$\begin{aligned}
S(\phi a, \phi a, a) &= S(\mu a, \mu a, \psi t) \\
&\leq k_1 \left[ \frac{S(\phi a, \phi a, \mu a).S(\phi t, \phi t, \psi t)}{S(\phi a, \phi a, \phi t) + S(\phi a, \phi a, \mu t) + S(\phi t, \phi t, \mu a)} \right] \\
&\qquad + k_2 S(\phi a, \phi a, \phi t) \\
&\leq k_1 \left[ \frac{S(\phi a, \phi a, \phi a).S(a, a, a)}{S(\phi a, \phi a, a) + S(\phi a, \phi a, a) + S(a, a, \phi a)} \right] + k_2 S(\phi a, \phi a, a) \\
&\leq k_2 \quad S(\phi a, \phi a, a),
\end{aligned}$$

leading to a contradiction, giving that $S(\phi a, \phi a, a) = 0$ implies $\phi a = a$.
Hence $\phi a = \psi a = \mu a = a$, showing that $a$ is a common fixed point of $\mu, \psi$ and $\phi$.
We now prove that the common fixed point is unique.
Suppose $a'(\neq a)$ is another common fixed point of $\mu, \psi$ and $\phi$.
That is $a' = \phi a' = \psi a' = \mu a'$.
We have

$$\begin{aligned}
S(a, a, a') &= S(\mu a, \mu a, \psi a') \\
&\leq k_1 \left[ \frac{S(\phi a, \phi a, \mu a).S(\phi a', \phi a', \psi a')}{S(\phi a, \phi a, \phi a') + S(\phi a, \phi a, \psi a') + S(\phi a', \phi a', \mu a)} \right] \\
&\qquad + k_2 S(\phi a, \phi a, \phi a') \\
&\leq k_1 \left[ \frac{S(a, a, a).S(a', a', a')}{S(a, a, a') + S(a, a, a') + S(a, a, a')} \right] + k_2 S(a, a, a') \\
&\leq k_2 S(a, a, a'),
\end{aligned}$$

which is a contradiction since $k_2 < 1$.
Therefore $S(a, a, a') = 0$ implies $a = a'$, Proving the uniqueness. □

**Corolary 3.1.** *Let P be a subset of a S-metric space $(Y, S)$. Suppose that $\phi, \mu$ are two selfmaps of P satisfy*

*(i)* $\mu(Y) \subseteq \phi(Y)$ *and* $(\phi(P), S)$ *is complete.*

*(ii)*

$$\begin{aligned}
&S(\mu y_1, \mu y_1, \mu y_2) \\
&\leq k_1 \{ \frac{S(\phi y_1, \phi y_1, \mu y_1).S(\phi y_2, \phi y_2, \mu y_2)}{S(\phi y_1, \pi y_1, \phi y_2) + S(\phi y_1, \phi y_1, \mu y_2) + S(\phi y_2, \phi y_2, \mu y_1)} \} \\
&\qquad + k_2 S(\phi y_1, \phi y_1, \phi y_2)
\end{aligned}$$

*for every* $y_1, y_2 \in P$ *and* $k_1, k_2 \in [0, 1)$ *with* $2k_1 + k_2 < 1$.

*(iii)* *The pair* $(\mu, \phi)$ *is weakly compatible.*
*Then* $\mu$ *and* $\phi$ *have a unique common fixed point.*

*Proof.* On taking $\psi = \mu$ in Theorem 3.1, the corollary follows. □

# 4 Conclusion

In this paper, a common fixed theorem for three weakly compatible selfmaps of a S-metric space is established with the aid of an associated sequence of three selfmaps.Moreover, we deduce a common fixed point theorem for two selfmaps. As S-metric space is a robust generalization of metric space, our theorem generalizes the theorem in literature.

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