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Combination of survival probabilities of the components in a system. An application to long-term financial valuation[1]

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Abstract. The Net Present Value (NPV) is a well-known method to value an investment project. Nevertheless, this methodology exhibits a serious problem when the used discounting function decreases very rapidly, especially in (very) long-term projects, because the future cash-flows are not significant in the expression of the NPV. For this reason, this paper introduces a methodology to correct the discounting function used for valuing. To do this, a new operation between discounting functions is defined by reducing the (cumulative) instantaneous discount rate corresponding of the valuing discounting function with another appropriate discounting function. The result is a new discounting function which can be more adequate to value this class of investment projects.

Keywords. Combination, discounting function, (cumulative) instantaneous discount rate, Net Present Value, investment project.

1. Introduction

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It is well-known that traditionally the exponential discounting has been used in the valuation of investment projects, as discounting function. But the main problem that exhibits this type of discount is the geometrical diminishing of its corresponding factors. In effect, the expression $(1+i)^{-t}$ decreases exponentially whereby future cash-flows, being very important, are not significant in the expression of the Net Present Value (NPV). This is the reason whereby our aim is to considerer a diminishing discount rate which would imply, at least, a decay of the corresponding (cumulative) instantaneous discount rate.

On the other hand, in a previous work, Cruz and Muñoz (2005 and 2007) introduced a new point of view of determining the social rate of discount and, more concretely, the discount function to be applied in the valuation of (very) long-term environmental and governmental projects. To do this, they started from the hazard rate of the system to which the project we are trying to value is addressed. In this way, if we are trying to value the construction of a public good (for example, a highway), the hazard rate corresponding to this construction along his useful life will supply us its survival probability (defined as the complement to the unit of the corresponding distribution function) which we will identify with the discounting function to be used in the valuation.

Thus, the instantaneous hazard rate of an investment is identified with the instantaneous discount rate corresponding to the discounting function necessary to value the project. As a consequence, the discounting function will be the survival probability of the system. More widely, the following table establishes the correspondence between several concepts from Finance (see, for example, Gil, 1993) and from Reliability Theory (see, for example, Barlow and Proschan, 1996).

Reliability Theory		**Finance**	
Survival probability (distribution tail)	$S(t) = 1 - F(t)$	Discounting function	$A(t)$
Instantaneous hazard rate	$h(t) = -\dfrac{\mathrm{d}S(t)/\mathrm{d}t}{S(t)}$	Instantaneous discount rate	$\delta(t) = -\dfrac{\mathrm{d}A(t)/\mathrm{d}t}{A(t)}$

Density function	$f(t) = -\frac{\mathrm{d}S(t)}{\mathrm{d}t}$	Cumulative inst. discount rate	$\nu(t) = -\frac{\mathrm{d}A(t)}{\mathrm{d}t}$
Conditional probability	$\frac{1-F(t+s)}{1-F(t)}$	Discounting factor	$\frac{A(t+s)}{A(t)}$

Table 1. Correspondence of concepts from Reliability Theory and Finance.

Starting from this methodology, we can obtain a (cumulative) instantaneous discount rate with two important advantages. The first one is that this magnitude is variable and the second one is that it can be diminishing. In this way, we agree with Harvey's (1986) position who proposes the hyperbolic or hyperbola-like discounting function and later (1994) defends variable discount rates.

Harvey (1994) examines "the reasonableness for public policy analysis of non-constant discounting method that, unlike constant discounting, can accord considerable importance to outcomes in the distant future". In his work, he proposes a method with positive discount rates that decrease and converge to zero as time converges to infinity.

The organization of this paper is as follows. In Section 2, we introduce a new algebraic operation between the survival probabilities of two components in a system. Taking into account Table 1, this is the same as define an algebraic operation between two discounting functions. Section 2 introduces a novel classification of discounting functions in singular and regular ones. Later, Section 4 presents a noteworthy application of Section 2 for the valuation of (very) long-term investment projects, avoiding the problems exhibited by a rapidly decreasing discounting function. Finally, Section 5 summarizes and concludes.

2. Combination of survival probabilities of the components in a system. Combination of discounting functions

Let us consider a structure composed by two independent components whose i-th component (i = 1, 2) has probability

$p_i(t) = 1 - F_i(t)$ of still being operative at time t. If $h(\mathbf{p})$ is the structure reliability function, where $\mathbf{p} = (p_1, p_2)$, and $S(t) = 1 - F(t)$ is the probability of the structure survival past time t, then since

$$S(t) = h(\mathbf{p}(t)),$$

we obtain:

$$-\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\partial h}{\partial p_1}\left(-\frac{\mathrm{d}p_1}{\mathrm{d}t}\right) + \frac{\partial h}{\partial p_2}\left(-\frac{\mathrm{d}p_2}{\mathrm{d}t}\right). \tag{1}$$

It is well-known that $-\dfrac{\mathrm{d}S}{\mathrm{d}t} := f(t)$ is the *density function* of the variable T describing the useful life of the system, and $-\dfrac{\mathrm{d}p_1}{\mathrm{d}t} := f_1(t)$ and $-\dfrac{\mathrm{d}p_2}{\mathrm{d}t} := f_2(t)$ are the *density* functions of variables T_1 and T_2 describing the useful life of components 1 and 2, respectively.

A noteworthy case is that in which the density function of the structure survival is the density function of component 1 but reduced by the effect of the survival probability of component 2, that is to say:

$$f(t) = f_1(t) \cdot p_2(t) = -\frac{\mathrm{d}p_1(t)}{\mathrm{d}t}\, p_2(t). \tag{2}$$

In this case,

$$\begin{aligned}
&S(t) := (p_1 \otimes p_2)(t) = 1 - \int_0^t -\frac{\mathrm{d}S}{\mathrm{d}x}\,\mathrm{d}x = \\
&= 1 - \int_0^t -\frac{\mathrm{d}p_1(x)}{\mathrm{d}x}\, p_2(x)\mathrm{d}x = \\
&1 - \int_0^t p_2(x)\left[-\mathrm{d}p_1(x)\right] = 1 - \int_0^t f_1(x)\, p_2(x)\mathrm{d}x.
\end{aligned} \tag{3}$$

that is, $-\mathrm{d}p_1(t)$, as an approximation of $-\Delta p_1 = -p_1(t+h) + p_1(t)$, is reduced by the survival probability of component 2. This new function $p_1 \otimes p_2$ will be called the *combination of survival probabilities* p_1 and p_2.

Taking into account Table 1, we can export this concept to Finance. Thus, the *combination* $A_1 \otimes A_2$ *of two discounting functions* A_1 and A_2:

$$A(t) := (A_1 \otimes A_2)(t) = 1 - \int_0^t -\frac{\mathrm{d}A}{\mathrm{d}x}\mathrm{d}x =$$
$$= 1 - \int_0^t -\frac{\mathrm{d}A_1(x)}{\mathrm{d}x} A_2(x)\mathrm{d}x =$$
$$1 - \int_0^t A_2(x)[-\mathrm{d}A_1(x)] = 1 - \int_0^t \nu_1(x) A_2(x)\mathrm{d}x \tag{4}$$

can be interpreted as a methodology to reduce the time perception of an objective discounting function (A_1) by the effect of another discounting function (A_2). This is because the more aged the individuals, the less time perception. In effect, a year of future time is not the same for a person *r* years old than a person *s* years old, being $r > s$. In this case, the time perception is greater for the second one.

In what follows and taking into account the aim of this paper, we will only refer to the combination of two discounting functions.

Example 1 Let us consider the combination of two simple discounting functions of parameters *d* and *d*′:

$$A(t) = 1 - d \cdot t\left(1 - d'\frac{t}{2}\right).$$

The following proposition supplies a preliminary basic inequality.

Proposition 1 $A_1(t) < A(t) < 1 - [1 - A_1(t)]A_2(t)$.

Proof. In effect, for the first inequality, as $0 < A_2(x) < 1$, it is verified that:

$$\int_0^t \nu_1(x) A_2(x)\mathrm{d}x < \int_0^t \nu_1(x)\mathrm{d}x$$

and so

$$A(t) = 1 - \int_0^t \nu_1(x) A_2(x)\mathrm{d}x > 1 - \int_0^t \nu_1(x)\mathrm{d}x = A_1(t).$$

For the second inequality, as function A_2 is strictly decreasing, take into account that:

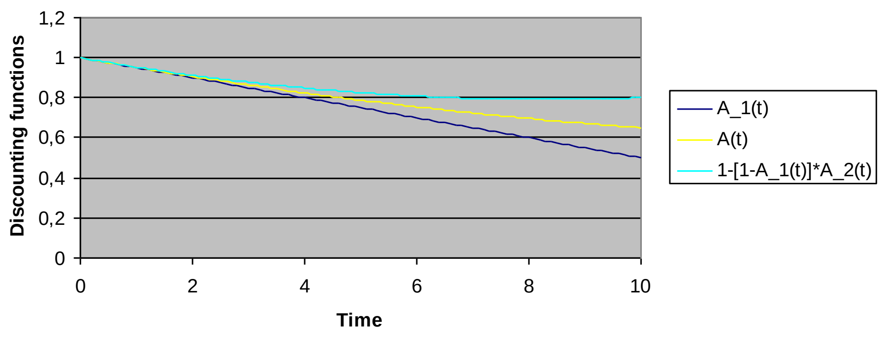
$$\int_0^t \nu_1(x) A_2(x)\mathrm{d}x > A_2(t)\int_0^t \nu_1(x)\mathrm{d}x = [1 - A_1(t)]A_2(t).$$

Thus,

$$A(t) = 1 - \int_0^t \nu_1(x) A_2(x)\mathrm{d}x < 1 - [1 - A_1(t)]A_2(t). \quad \square$$

A graphic representation of the discounting function obtained in Example 1 for $d = 0.05$ and $d' = 0.06$, and a confirmation of the result deduced in Proposition 1, can be seen in Figure 1.

Figure 1. Confirming Proposition 1

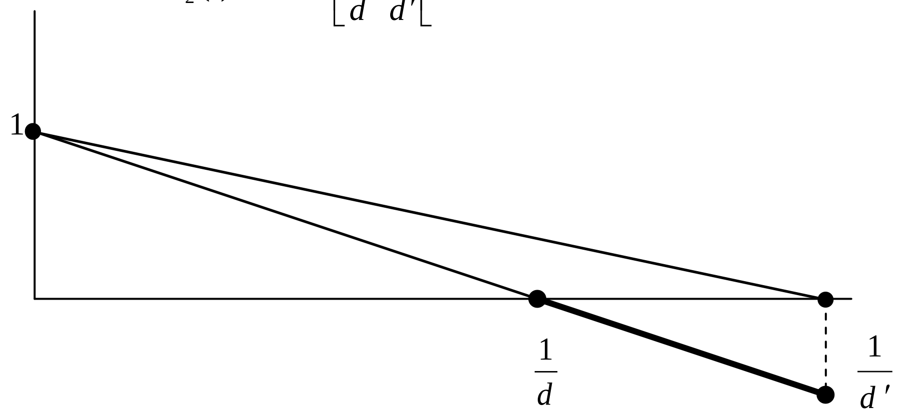


With respect to the temporal domain, $D(t)$, of the new discounting function, two cases can occur ($D_1(t)$ and $D_2(t)$ are the time discounting domains of the discounting functions A_1 and A_2, respectively):

- If $D_2(t) \subseteq D_1(t)$, then $D(t) = D_2(t)$.
- If $D_1(t) \subset D_2(t)$, then $D_1(t) \subseteq D(t) \subseteq D_2(t)$, because, if $D_1(t) = [0, t_1]$, there can exist a non-empty interval $[t_1, t_2[\subseteq D_2(t) - D_1(t)$ where $A_1 < 0$ and $\frac{\mathrm{d}A_1}{\mathrm{d}t} < 0$. In this case, $D(t) = D_1(t) \cup [t_1, t_2[$. Observe that eventually $D(t)$ can coincide with $D_2(t)$.

In Example 1, $D_1(t) = \left[0, \frac{1}{d}\right[$ and $D_2(t) = \left[0, \frac{1}{d'}\right[$. Consequently, two cases can occur:

- If $d \le d'$, $\frac{1}{d'} \le \frac{1}{d}$ and so $D_2(t) \subseteq D_1(t)$. Thus, $D(t) = D_2(t) = \left[0, \frac{1}{d'}\right[$.
- If $d > d'$, $\frac{1}{d} < \frac{1}{d'}$ and so $D_1(t) \subset D_2(t)$. As A_1 is decreasing and $A_2(t) > 0$ in $\left[\frac{1}{d}, \frac{1}{d'}\right[$:



then there exists a t_2 such that $D(t) = D_1(t) \cup \left[\frac{1}{d}, t_2\right[$. To calculate t_2, we have to solve the equation:

$$dd'\frac{t^2}{2} - dt + 1 = 0,$$

which only has a solution if and only if $d' < \frac{d}{2}$. In this case the obtained solution is $t = \frac{d \pm \sqrt{d^2 - 2dd'}}{dd'} = \frac{1 \pm \sqrt{1 - 2\frac{d'}{d}}}{d'}$, from where $t_2 = \frac{1 - \sqrt{1 - 2\frac{d'}{d}}}{d'}$, which obviously lesser than $\frac{1}{d'}$. On the other hand, writing the solution as $t_2 = \frac{d - \sqrt{(d-d')^2 - d'^2}}{dd'}$, we can show that $t_2 > \frac{d - (d - d')}{dd'} = \frac{1}{d}$.

Definition 1 Let A_1 and A_2 be two discounting functions. The *ordinary product* of both functions, denoted by $A_1 \cdot A_2$, is defined in the following way:

$$(A_1 \cdot A_2)(t) = A_1(t) \cdot A_2(t).$$

Observe that this algebraic operation reflects the “multiplicative” superposition of the effects due to both discounting functions over a certain temporal interval.

Definition 2 Let A_1 and A_2 be two discounting functions. The *reduced sum* of both functions, denoted by $A_1 \oplus A_2$, is defined in the following way:

$$(A_1 \oplus A_2)(t) = A_1(t) + A_2(t) - 1.$$

Once defined these algebraic operations, we can enunciate the following

Proposition 2 $(A_1 \otimes A_2) \oplus (A_1 \otimes A_2) = A_1 \cdot A_2$.

Proof. In effect, by calculating the integral $\int_0^t \nu_1(x) A_2(x) \mathrm{d}x$ by parts, we have:

$$\int_0^t \nu_1(x) A_2(x) \mathrm{d}x = -A_1(t) \cdot A_2(t) + 1 - \int_0^t A_1(x) \nu_2(x) \mathrm{d}x,$$

from where we can easily deduce the required equality. □

The following theorem relates the convexity of discounting functions A and A_1.

Theorem 1 If A_1 is convex, then $A = A_1 \otimes A_2$ is also convex, independently of the convexity or concavity of A_2.

Proof. From Equation (4), $\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \frac{\mathrm{d}A_1(t)}{\mathrm{d}t} A_2(t)$. Differentiating again with respect to *x*:

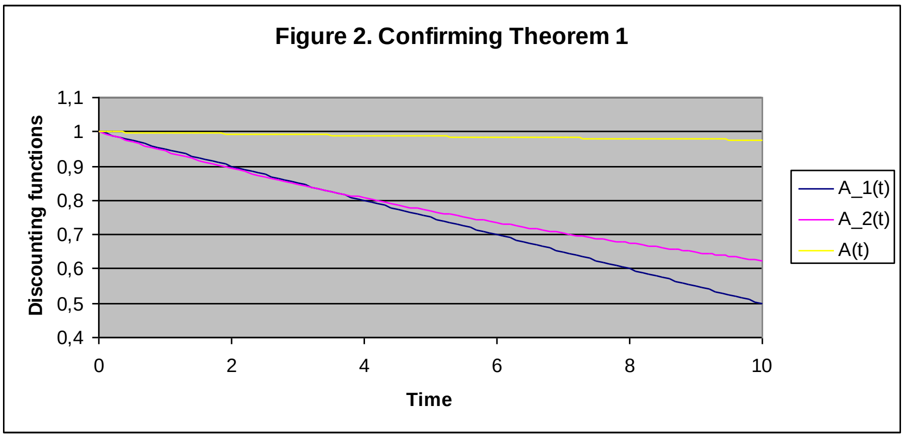
$$\frac{\mathrm{d}^2 S(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 A_1(t)}{\mathrm{d}t^2} A_2(t) + \frac{\mathrm{d}A_1(t)}{\mathrm{d}t} \frac{\mathrm{d}A_2(t)}{\mathrm{d}t}.$$

As p_1 is convex, $\frac{\mathrm{d}^2 A_1(t)}{\mathrm{d}t^2} > 0$. Moreover, as $\frac{\mathrm{d}A_1(t)}{\mathrm{d}t}$ and $\frac{\mathrm{d}A_2(t)}{\mathrm{d}t}$ are negative, and obviously $A_2(t) > 0$, then *S* is convex. □

Example 2 Let us consider the combination of the simple discounting function of parameters d and the hyperbolic discounting of parameter i:

$$A(t) = 1 - d \cdot \ln(1 + i \cdot t).$$

A graphic representation of the discounting function obtained in Example 2 for $d = 0.05$ and $i = 0.06$, and a confirmation of the result deduced in Theorem 1, can be seen in Figure 2.



Obviously, the operation $\otimes$ does not verify the commutative property, but we can easily show the following proposition. Observe that there exists an "exchange by quotient" between the cumulative instantaneous discount rates of the two combinations of discounting functions towards the instantaneous discount rates corresponding to components 1 and 2.

Proposition 2 The following equality holds:

$$\frac{\mathrm{d}(A_1 \otimes A_2)(t)}{\mathrm{d}(A_2 \otimes A_1)(t)} = \frac{\delta_1(t)}{\delta_2(t)}.$$

Proof. It is obvious taking into account that $\frac{\mathrm{d}(A_1 \otimes A_2)(t)}{\mathrm{d}t} = \frac{\mathrm{d}A_1(t)}{\mathrm{d}t} A_2(t)$, $\frac{\mathrm{d}(A_2 \otimes A_1)(t)}{\mathrm{d}t} = \frac{\mathrm{d}A_2(t)}{\mathrm{d}t} A_1(t)$, $\delta_1(t) = -\frac{\mathrm{d}A_1(t)/\mathrm{d}t}{A_1(t)}$ and $\delta_2(t) = -\frac{\mathrm{d}A_2(t)/\mathrm{d}t}{A_2(t)}$. □

We can check the result obtained in Proposition 2 with the following example.

Example 3 Combination of two exponential discounting functions of parameters i and i' ($i < i'$):

$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \ln(1+i)\left[(1+i)(1+i')\right]^{-t}$ and $\frac{\mathrm{d}A'(t)}{\mathrm{d}t} = \ln(1+i')\left[(1+i)(1+i')\right]^{-t}$.

Therefore, we can obviously check that:

$$\frac{\mathrm{d}(A_1 \otimes A_2)(t)}{\mathrm{d}(A_2 \otimes A_1)(t)} = \frac{\ln(1+i)}{\ln(1+i')} = \frac{\delta_1(t)}{\delta_2(t)}.$$

Proposition 3 The operation $\otimes$ does not verify the commutative property except for equal elements. Moreover, it is cancellative on the left and on the right.

Proof. In effect, if $(A_1 \otimes A_2)(t) = (A_2 \otimes A_1)(t)$, then $\frac{\mathrm{d}A_1(t)}{\mathrm{d}t} A_2(t) = \frac{\mathrm{d}A_2(t)}{\mathrm{d}t} A_1(t)$ and, consequently, $\delta_1(t) = -\frac{\mathrm{d}A_1(t)}{\mathrm{d}t} \cdot \frac{1}{A_1(t)} = -\frac{\mathrm{d}A_2(t)}{\mathrm{d}t} \cdot \frac{1}{A_2(t)} = \delta_2(t)$. Thus $A_1(t) = A_2(t)$. On the other hand, if $(A_1 \otimes A_2)(t) = (A_1 \otimes A_3)(t)$, by definition, $\frac{\mathrm{d}A_1(t)}{\mathrm{d}t} A_2(t) = \frac{\mathrm{d}A_1(t)}{\mathrm{d}t} A_3(t)$ and then $A_2(t) = A_3(t)$. Finally, if

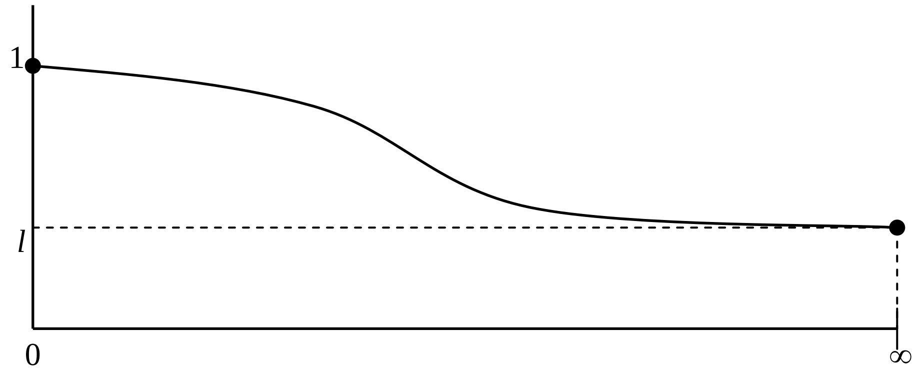
$(p_1 \otimes p_3)(t) = (p_2 \otimes p_3)(t)$, by definition, $\frac{\mathrm{d}A_1(t)}{\mathrm{d}t} A_3(t) = \frac{\mathrm{d}A_2(t)}{\mathrm{d}t} A_3(t)$ and then $A_1(t) = A_2(t)$. □

3. Singular and regular discounting functions

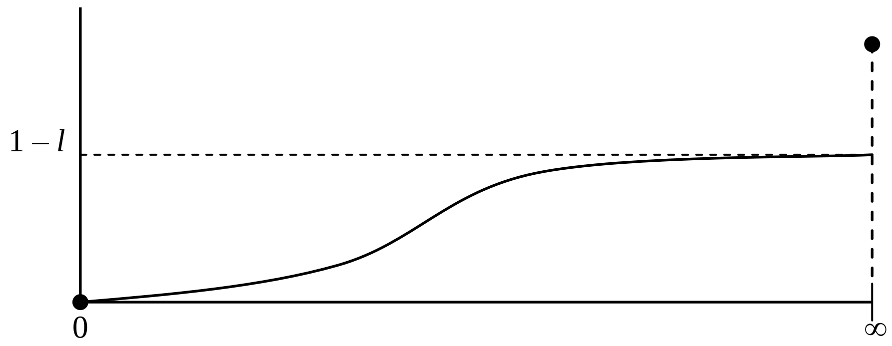
Definition 3 (Maravall, 1970) A discounting function $A(t)$ is said to be *singular* if $\lim_{t\to\infty} A(t) \neq 0$ or there exists a real number t_0 such that $A(t_0) \neq 0$. Otherwise, $A(t)$ is said to be *regular*.

Example 4 The discounting function is $A(t) = \frac{1+i\cdot t}{1+j\cdot t}$, where $i < j$, is singular because, obviously, $\lim_{t\to\infty} A(t) = \frac{i}{j}$. Obviously, hyperbolic discounting is regular.

A singular discounting function is a peculiar discounting function which has a horizontal asymptote at $y = l$, where l can be interpreted as the mass of probability at infinity of the corresponding distribution function. Representing this function in the extended real numbers:



and so its corresponding distribution function:



Definition 3 provides a *classification of discounting functions*:

1. *Singular discounting functions with bounded domain*: $D(t) = [0, t_0[$ and $A(t_0) \neq 0$. More specifically, $0 < A(t_0) < 1$
2. *Regular discounting functions with bounded domain*: $D(t) = [0, t_0[$ and $A(t_0) = 0$.
3. *Singular discounting functions*: $D(t) = [0, +\infty[$ and $\lim_{t \to \infty} A(t) \neq 0$. More specifically, $0 < \lim_{t \to \infty} A(t) < 1$.
4. *Regular discounting functions*: $D(t) = [0, +\infty[$ and $\lim_{t \to \infty} A(t) = 0$.

It is possible to provide some results on all possible combinations of different class of discounting functions. For instance, it can be shown that the combination of two singular discounting functions with bounded domain is also singular with bounded domain (see Example 1) and that the combination of a singular and a regular discounting function with bounded domain is singular with bounded domain. Finally, next examples show the result of combining of some well-known discounting function.

Example 5 Combination of a hyperbolic discounting function of parameter *i* and a simple discounting of parameter *d*:

$$A(t) = 1 - \frac{1}{i}\left(1 - \frac{1}{1+it}\right) - \frac{d}{i^2}\left[1 - \ln(1+it) - \frac{1}{1+it}\right].$$

Example 6 Combination of two hyperbolic discounting functions of parameters i and j:

$$A(t) = 1 + \frac{j}{(i-j)^2} \ln \frac{(1+it)^i}{(1+jt)^j} - \frac{i}{i-j}\left(1 - \frac{1}{1+it}\right) \text{ (singular).}$$

Example 7 Combination of a simple discounting function of parameter d and an exponential discounting function of parameter k:

$$A(t) = 1 - \frac{d}{k}\left(1 - e^{-kt}\right) \text{ (singular).}$$

Example 8 Combination of an exponential discounting function of parameter k and a simple discounting function of parameter d:

$$A(t) = (1 - dt)e^{-kt} + \frac{d}{k}\left(1 - e^{-kt}\right) \text{ (singular).}$$

4. The combination of discounting functions in the valuation of governmental projects

Consider the case in which a government must decide if a (very) long-term investment project is feasible. It is well-known that, to valuate this project, the most important discounting function to be used in the net present value (NPV) formula is the exponential one $A_1(t) = (1+i)^{-t}$, being i the technical interest rate:

$$NPV = -A + \sum_{k=1}^{n} CF_k \cdot A_1(k),$$

where:

- NPV is the net present value of the project;
- A is the initial payment of the project;
- n is the useful life of the project;
- CF_k is the *k*-th cash-flow corresponding to the project.

Assume that the survival probability of the system or the perception time of the population is described by the discounting function $A_2(t)$. In this case, it could be convenient to reinforce the first discounting function with the aim of preserve the future cash-flows. Thus, the formula to be employed would be:

$$NPV = -A + \sum_{k=1}^{n} CF_k \cdot (A_1 \otimes A_2)(k),$$

leading to smaller discount rates, more appropriate to value the aforementioned governmental projects.

5. Conclusion

In (very) long-term project appraisal (for example, governmental and environmental projects), the exponential discounting function has been traditionally used to update the future cash-flows at the present moment. Despite its generalized use, exponential discounting presents an obvious problem: the geometric diminishing of the actualization factors "almost annihilates" the most distant cash-flows. Therefore, it is necessary to increase the discounting function with the aim of reaching a higher presence of the further cash-flows.

To do this, there are several procedures. The methodology used in this paper is based on the idea of a diminishing perception of future time. Indeed, empirical researches show that for most people the larger the age of the person, the shorter the time periods. Thus, the "perceived" discounted amounts must be lesser and this fact must be reflected in the mathematical expression of the "true" discounting function. In this work, the reduction in the discounted values can be reached with another discounting function through the so-called combination of discounting functions.

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