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Common Fixed Point for a Contractive Mapping via $(\alpha,\ \beta,\ \psi)$-Admissibility in b-metric space

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Abstract

In this paper, we establish the concept of a common fixed point theorem for new type of generalized contractive mappings. Furthermore, we employ our main result to shows a common fixed point theorem for a pair of self-mappings (R, S) in b-metric space via (α, β, ψ)-admissibility type contractive condition. An example is also given to verify the main result.
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1 Introduction

In the last fifty years, fixed point theories lie in finding and proving the uniqueness of solutions for many questions of Applied Sciences such as Physics, Chemistry, Economics, and Engineering. In 1922, Stefan Banach [S.Banach [1922]] proved a fixed point theorem for contractive mappings in complete metric spaces. In 1969, Nadler [Nadler [1969]] introduced the concept of multi-value function. Later, Czerwik [Czerwik [1993]] and Bakhtin [Bakhtin [1989]] initiate the concept of b-metrics metric space. Khan [Khan et al. [1984]] introduced the altering distance mapping to formulate a new contractive condition in fixed point theory in order to extend the Banach fixed point theorem to new forms. For some extension to the Banach contraction theorem. Recently, Abodayeh et al. [Abodayeh et al. [2017]] introduced a new notion, named almost perfect function, to formulate new contractive conditions to modify and extend some fixed point theorems known in the literature. Now, we mention the notions of altering distance function and almost perfect function.

2 Preliminaries

Definition 2.1 (Khan et al. [1984]). *A self-function ψ on $\mathbb{R}^+ \cup \{0\}$ is called an altering distance function if ψ satisfies the following conditions:*

(1) $\psi(s) = 0 \iff s = 0$.

(2) ψ is a nondecreasing and continuous function.

Definition 2.2 (Abodayeh et al. [2017]). *A nondecreasing self-function ψ on $\mathbb{R}^+ \cup \{0\}$ is called an almost perfect function if ψ satisfies the following conditions:*

(1) $\psi(s) = 0 \iff s = 0$.

(2) If for all sequence (s_n) in $\mathbb{R}^+ \cup \{0\}$ with $\psi(s_n) \to 0$ it holds $s_n \to 0$.

Definition 2.3 (Samet et al. [2012]). *Let R be a self-mapping on X and $\alpha : X \times X \to \mathbb{R}^+ \cup \{0\}$ be a function. Then, R is called α-admissible if for all $v,\ w \in X$ with $\alpha(v,\ w) \geq 1$ it holds $\alpha(Rv,\ Rw) \geq 1$.*

The definition of triangular α-admissibility for a single mapping

Definition 2.4 (Karupinar et al. [2013]). *Let R be a self-mapping on X and $\alpha : X \times X \to \mathbb{R}^+ \cup \{0\}$. Then, we call R triangular α-admissible if*

(1) R is α-admissible; and

(2) For all $v,\ w,\ u \in X$ with $\alpha(v,\ w) \geq 1$ and $\alpha(w,\ u) \geq 1$ it holds $\alpha(v,\ w) \geq 1$.

Definition 2.5 (Abdeljawad [2013]). *Let R and S be two self mappings on X and $\alpha : X \times X \to \mathbb{R}^+ \cup \{0\}$ be a function. Then, the pair $(R,\ S)$ is called α-admissible if $z,\ w \in X$ and $\alpha(z,\ w) \geq 1$ imply $\alpha(Rz,\ Sw) \geq 1$ and $\alpha(Sz,\ Rw) \geq 1$.*

Definition 2.6 (Hussain et al. [2014]). *Let D_b be a metric on a set X and $\alpha,\ \beta : X \times X \to \mathbb{R}^+ \cup \{0\}$ be functions. Then, X is called $\alpha,\ \beta$-complete if and only if $\{x_n\}$ is a Cauchy sequence in X and $\alpha(x_n,\ x_{n+1}) \geq \beta(x_n,\ x_{n+1})$ for all $n \in N$ imply (x_n) converges to some $x \in X$.*

Definition 2.7 (Hussain et al. [2014]). *Let D_b be a metric on a set X and $\alpha,\ \eta : X \times X \to \mathbb{R}^+ \cup \{0\}$ be functions. A self-mapping S on X is called $\alpha,\ \beta$-continuous if $\{x_n\}$ is a sequence in X, $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,\ x_{n+1}) \geq \beta(x_n,\ x_{n+1})$ for all $n \in N$ imply $Sx_n \to Sx$ as $n \to \infty$.*

Definition 2.8 (Mehemet and Kiziltunc [2013]). *Let X be a non-empty set and let $s \geq 1$ be a given real number. A function $D_b : X \times X \to \mathbb{R}^+ \cup \{0\}$ is called a b-metric provide that, for all $x,\ y,\ z \in X$,*

(1) $D_b(x,\ y) = 0\ if\ and\ only\ if\ x = y$ (non-negative axiom)

(2) $D_b(x,\ y) = D_b(y,\ x)$ (symmetric axiom)

(3) $D_b(x,z) \leq s[D_b(x,\ y) + D_b(y,\ z)]$. (s-Triangular inequality).

A pair $(X,\ D_b)$ is called a b-metric space.

Definition 2.9. *Let $R, S,$ be two self-mappings on the set X and $\alpha,\ \beta : X \times X \to \mathbb{R}^+ \cup \{0\}$ be functions. We say that $(R,\ S)$ is a pair of $(\alpha,\ \beta)$-admissibility if $z,\ w \in X$ and $\alpha(z,\ w) \geq \beta(z,\ w)$ imply $\alpha(Rz,\ Sw) \geq \beta(Rz,\ Sw)$ and $\alpha(Sz,\ Rw) \geq \beta(Sz,\ Rw)$.*

Example 2.1. *Define self-mappings R and S on a set of real numbers by $Ru = u^2$ and*

$$Su = \begin{cases} -u^2, & if\ u < 0; \\ u^2, & if\ u \geq 0. \end{cases}$$

Additionally, define $\alpha, \beta : X \times X \to \mathbb{R}^+ \cup \{0\}$ via $\alpha(u, v) = e^{u+v}$ and $\beta(u, v) = e^u$. Then, (R, S) is a pair of (α, β)-admissibility.

3 Main Results

Definition 3.1. *Let ψ be a nondecreasing function on $\mathbb{R}^+ \cup \{0\}$. We call ψ a perfect control function if the following conditions hold:*

(i) $\psi(t) = 0 \iff t = 0.$

(ii) If (t_n) is a sequence in $\mathbb{R}^+ \cup \{0\}$ and $\psi(t_n) \to 0$ as $n \to +\infty$ implies $t_n \to 0$ as $n \to +\infty$.

(iii) $\psi(u+v) \leq \psi(u) + \psi(v)$ *for all* $u,\ v \in \mathbb{R}^+ \cup \{0\}$ *.*

(iv) $\psi^n(\lambda x) = \lambda^n \psi(x)$.

Definition 3.2. *Let (X, D_b) be a b-metric space with constant $s \geq 1$. Let $R,\ S$ be two self-mappings on X, ψ be a perfect self-mapping on $\mathbb{R}^+ \cup \{0\}$, $\alpha,\ \beta\ :\ X \times X \to \mathbb{R}^+ \cup \{0\}$ be functions. We say that the pair $(R,\ S)$ is an $(\alpha,\ \beta,\ \psi)$-Admissibility type contraction if there exists $\lambda \in [0,\ 1)$ such that $z,\ w \in X$ and $\alpha(z,\ w) \geq \beta(z,\ w)$ imply*

$$\begin{aligned} \psi(D_b(Rz, Sw)) \quad &\leq \lambda\psi(D_b(z,w)) + \lambda\psi(D_b(z,Rz)) + \lambda\psi(D_b(w,Sw)) \\ &+\lambda\psi(D_b(w,Rz)) + \lambda\psi(D_b(z,Sw)) \end{aligned} \tag{1}$$

and

$$\begin{aligned} \psi(D_b(Sz, Rw)) \quad &\leq \lambda\psi(D_b(z,w)) + \lambda\psi(D_b(z,Sz)) + \lambda\psi(D_b(w,Rw)) \\ &+\lambda\psi(D_b(w,Sz)) + \lambda\psi(D_b(z,Rw)) \end{aligned} \tag{2}$$

Theorem 3.1. *Let (X, D_b) be a b-metric space with constant $s \geq 1$. Let $\alpha,\ \beta\ :\ X \times X \to \mathbb{R}^+ \cup \{0\}$ be function and (R, S) be a self-mappings on X. Assume following conditions:*

(i) (X, D_b) is an α, β-complete b-metric space.

(ii) R and S are α, β-continuous.

(iii) (R, S) is pair of (α, β)-admissibility.

(iv) If v, w, z are in X, with $\alpha(v, w) \geq \beta(v, w)$ and $\alpha(w, z) \geq \beta(w, z)$, then $\alpha(v, z) \geq \beta(v, z)$.

(v) There exists $x_0 \in X$ such that $\alpha(Rx_0, SRx_0) \geq \beta(Rx_0, SRx_0)$ and $\alpha(SRx_0, Rx_0) \geq \beta(SRx_0, Rx_0)$.

Then R and S have a common fixed point.

Proof. *In view of condition (v) we start with $x_0 \in X$ in such away that $\alpha(Rx_0, SRx_0) \geq \beta(Rx_0, SRx_0)$ and $\alpha(SRx_0, Rx_0) \geq \beta(SRx_0, Rx_0)$. Now, let $x_1 = Rx_0$ and $x_2 = Sx_1$. Then $\alpha(x_0, x_1) \geq \beta(x_0, x_1)$ and $\alpha(x_1, x_0) \geq \beta(x_1, x_0)$. In view of condition (iii), we have*

$$\alpha(x_1, x_2) = \alpha(Rx_0, Sx_1) \quad \geq \beta(Rx_0, Sx_1) = \beta(x_1, x_2)$$

and

$$\alpha(x_2, x_1) = \alpha(Sx_1, Rx_0) \quad \geq \beta(Sx_1, Rx_0) = \beta(x_2, x_1)$$

Again we put $x_3 = Sx_2$. Then condition (iii) implies that

$$\alpha(x_2, x_3) = \alpha(Sx_1, Rx_2) \geq \beta(Sx_1, Rx_2) = \beta(x_2, x_3)$$

and

$$\alpha(x_3, x_2) = \alpha(Rx_2, Sx_1) \geq \beta(Rx_2, Sx_1) = \beta(x_3, x_2)$$

Putting $x_4 = Sx_3$ and referring to condition (iii) we conclude

$$\alpha(x_3, x_4) = \alpha(Rx_2, Sx_3) \geq \beta(Rx_2, Sx_3) = \beta(x_3, x_4)$$

and

$$\alpha(x_4, x_3) = \alpha(Sx_3, Rx_2) \geq \beta(Sx_3, Rx_2) = \beta(x_4, x_3)$$

Continuing in the same manner, we contract a sequence (x_n) in X with $x_{2n+1} = Rx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ such that

$$\alpha(x_n, x_{n+1}) \quad \geq \beta(x_n, x_{n+1}) \quad \forall\, n \in \mathbb{N}$$

and

$$\alpha(x_{n+1}, x_n) \quad \geq \beta(x_{n+1}, x_n) \quad \forall\, n \in \mathbb{N}$$

From condition (iv) we see that

$$\alpha(x_n, x_m) \quad \geq \beta(x_n, x_m) \qquad \forall\, n, m \in \mathbb{N}$$

If there exists $q \in \mathbb{N}$ such that $x_{2q} = x_{2q+1}$, then $x_{2q} = Rx_{2q}$ and hence R has a fixed point. From contractive condition (1), we have

$$\begin{aligned}
&\psi(D_b(x_{2q+1}, x_{2q+2})) = \psi(D_b(Rx_{2q}, Sx_{2q+1})) \\
&\leq \psi\left(\begin{matrix} \lambda(D_b(x_{2q}, x_{2q+1})) + \lambda(D_b(x_{2q}, Rx_{2q})) + \lambda(D_b(x_{2q+1}, Sx_{2q+1})) \\ +\lambda(D_b(x_{2q+1}, Rx_{2q})) + \lambda(D_b(x_{2q}, Sx_{2q+1})) \end{matrix}\right) \\
&\leq \psi\begin{pmatrix} 2\lambda(D_b(x_{2q}, x_{2q+1})) + \lambda(D_b(x_{2q+1}, x_{2q+2})) \\ +\lambda s[(D_b(x_{2q}, x_{2q+1}) + D_b(x_{2q+1}, x_{2q+2}))] \end{pmatrix} \\
&\leq \psi\begin{pmatrix} \lambda(2+s)(D_b(x_{2q}, x_{2q+1})) \\ +\lambda(1+s)(D_b(x_{2q+1}, x_{2q+2})) \end{pmatrix}
\end{aligned}$$

$$\leq \psi\left(\tfrac{\lambda(2+s)}{1-\lambda(1+s)}(D_b(x_{2q}, x_{2q+1}))\right) \tag{3}$$

The last inequality is correct only if $\psi(\frac{\lambda(2+s)}{1-\lambda(1+s)}(D_b(x_{2q}, x_{2q+1}))) = 0$. *The properties of* ψ *and* D_b *imply that* $x_{2q+1} = x_{2q+2}$. *Hence,* $x_{2q} = Rx_{2q} = Sx_{2q}$. *Thus,* R *and* S *have a common fixed point of* R *and* S. *If there exists* $q \in \mathbb{N}$ *such that* $x_{2q+1} = x_{2q+2}$ *then* $x_{2q+1} = Tx_{2q+1}$ *and hence* S *has a fixed point.*
From contractive condition (2), we have

$$\psi(D_b(x_{2q+2}, x_{2q+3})) = \psi(D_b(Sx_{2q+1}, Rx_{2q+2}))$$

$$\leq \psi\begin{pmatrix}\lambda(D_b(x_{2q+1}, x_{2q+2})) + \lambda(D_b(x_{2q+1}, Sx_{2q+1})) + \lambda(D_b(x_{2q+2}, Rx_{2q+2}))\\ +\lambda(D_b(x_{2q+2}, Sx_{2q+1})) + \lambda(D_b(x_{2q+1}, Rx_{2q+2}))\end{pmatrix}$$

$$\leq \psi\begin{pmatrix}2\lambda(D_b(x_{2q+1}, x_{2q+2})) + \lambda(D_b(x_{2q+2}, x_{2q+3}))\\ +\lambda s[(D_b(x_{2q+1}, x_{2q+2}) + D_b(x_{2q+2}, x_{2q+3}))]\end{pmatrix}$$

$$\leq \psi\begin{pmatrix}\lambda(2+s)(D_b(x_{2q+1}, x_{2q+2}))\\ +\lambda(1+s)(D_b(x_{2q+2}, x_{2q+3}))\end{pmatrix}$$

$$\leq \psi\left(\tfrac{\lambda(2+s)}{1-\lambda(1+s)}(D_b(x_{2q+1}, x_{2q+2}))\right) \tag{4}$$

The last inequality is correct only if $\psi(\frac{\lambda(2+s)}{1-\lambda(1+s)}(D_b(x_{2q+1}, x_{2q+2}))) = 0$. *The properties of* ψ *and* D_b *imply that* $x_{2q+2} = x_{2q+3}$. *Hence,* $x_{2q+1} = Rx_{2q+1} = Sx_{2q+1}$. *Thus,* R *and* S *have a common fixed point of* R *and* S.
Now, assume that $x_n \neq x_{n+1}\ \forall\, n \in \mathbb{N}$. *For* $n \in \mathbb{N} \cup \{0\}$, *we get*

$$\psi(D_b(x_{2n+1}, x_{2n+2})) = \psi(D_b(Rx_{2n}, Sx_{2n+1}))$$

$$\leq \psi\begin{pmatrix}\lambda(D_b(x_{2n}, x_{2n+1})) + \lambda(D_b(x_{2n}, Rx_{2n})) + \lambda(D_b(x_{2n+1}, Sx_{2n+1}))\\ +\lambda(D_b(x_{2n+1}, Rx_{2n})) + \lambda(D_b(x_{2n}, Sx_{2n+1}))\end{pmatrix}$$

$$\leq \psi\begin{pmatrix}2\lambda(D_b(x_{2n}, x_{2n+1})) + \lambda(D_b(x_{2n+1}, x_{2n+2}))\\ +\lambda s[(D_b(x_{2n}, x_{2n+1}) + D_b(x_{2n+1}, x_{2n+2}))]\end{pmatrix}$$

$$\leq \psi\begin{pmatrix}\lambda(2+s)(D_b(x_{2n}, x_{2n+1}))\\ +\lambda(1+s)(D_b(x_{2n+1}, x_{2n+2}))\end{pmatrix}$$

$$\leq \psi\left(\tfrac{\lambda(2+s)}{1-\lambda(1+s)}(D_b(x_{2n}, x_{2n+1}))\right) \tag{5}$$

Let $[\delta = \frac{\lambda(2+s)}{1-\lambda(1+s)}]$. *Hence*

$$\psi(D_b(x_{2n+1}, x_{2n+2})) \leq \psi(\delta(D_b(x_{2n}, x_{2n+1})))$$

Using argument similar to the above, we may show that

$$\begin{aligned}
&\psi(D_b(x_{2n},x_{2n+1}))=\psi(D_b(Sx_{2n-1},Rx_{2n}))\\
&\leq \psi\begin{pmatrix}\lambda(D_b(x_{2n-1},x_{2n}))+\lambda(D_b(x_{2n-1},Sx_{2n-1}))+\lambda(D_b(x_{2n},Rx_{2n}))\\ +\lambda(D_b(x_{2n},Sx_{2n-1}))+\lambda(D_b(x_{2n-1},Rx_{2n}))\end{pmatrix}\\
&\leq \psi\begin{pmatrix}2\lambda(D_b(x_{2n-1},x_{2n}))+\lambda(D_b(x_{2n},x_{2n+1}))\\ +\lambda s[(D_b(x_{2n-1},x_{2n})+D_b(x_{2n},x_{2n+1}))]\end{pmatrix}\\
&\leq \psi\begin{pmatrix}\lambda(2+s)(D_b(x_{2n-1},x_{2n}))\\ +\lambda(1+s)(D_b(x_{2n},x_{2n+1}))\end{pmatrix}\\
&\leq \psi\left(\tfrac{\lambda(2+s)}{1-\lambda(1+s)}(D_b(x_{2n-1},x_{2n})))\right)\\
&\leq \psi\left((\delta(D_b(x_{2n-1},x_{2n})))\right) \qquad (6)
\end{aligned}$$

Combining equation (5) and (6) together, we reach

$$\begin{aligned}
&\psi(D_b(x_n,x_{n+1}))=\psi(D_b(Sx_{n-1},Rx_n))\\
&\leq \psi\begin{pmatrix}\lambda(D_b(x_{n-1},x_n))+\lambda(D_b(x_{n-1},Sx_{n-1}))+\lambda(D_b(x_n,Rx_n))\\ +\lambda(D_b(x_n,Sx_{n-1}))+\lambda(D_b(x_{n-1},Rx_n))\end{pmatrix}\\
&\leq \psi\begin{pmatrix}2\lambda(D_b(x_{n-1},x_n))+\lambda(D_b(x_n,x_{n+1}))\\ +\lambda s[(D_b(x_{n-1},x_n)+D_b(x_n,x_{n+1}))]\end{pmatrix}\\
&\leq \psi\begin{pmatrix}\lambda(2+s)(D_b(x_{n-1},x_n))\\ +\lambda(1+s)(D_b(x_n,x_{n+1}))\end{pmatrix}\\
&\leq \psi\left(\tfrac{\lambda(2+s)}{1-\lambda(1+s)}(D_b(x_{n-1},x_n))\right)\\
&\leq \psi\left(\delta(D_b(x_{n-1},x_n))\right) \qquad (7)
\end{aligned}$$

By recurring equation (7) n-times, we deduce

$$\begin{aligned}
\psi(D_b(x_n,x_{n+1})) &\leq \psi(\delta(D_b(x_{n-1},x_n)))\\
&\leq \delta\psi(D_b(x_{n-2},x_{n-1}))\\
&\leq \delta(\delta\psi(D_b(x_{n-2},x_{n-1})))\\
&= \delta^2\psi(D_b(x_{n-2},x_{n-1}))\\
&\ \vdots
\end{aligned}$$

$$\leq \delta^n \psi(D_b(x_0, x_1)). \tag{8}$$

On allowing $n \to \infty$ in equation (8), we get

$$\lim_{n\to+\infty} \psi(D_b(x_n, x_{n+1})) = 0 \tag{9}$$

The properties of ψ implies that

$$\lim_{n\to+\infty} D_b(x_n, x_{n+1}) = 0 \tag{10}$$

We intend to prove that (x_n) is Cauchy sequence in X, take $n, m \in \mathbb{N}$ with $m > n$. We divide the proof into four cases:
Case 1: *n is an odd integer and m is an even integer. Therefore, there exists $t \in \mathbb{N}$ and an odd integer h such that $n = 2t+1$ and $m = 2t+1+h$. Since $\alpha(x_n, x_m) \geq \beta(x_n, x_m)$, we have*

$$\psi(D_b(x_n, x_m)) = \psi(D_b(x_{2t+1}, x_{2t+1+h}))$$

$$= \psi(D_b(Rx_{2t}, Sx_{(2t+h)}))$$

$$\leq \psi\begin{pmatrix} \lambda(D_b(x_{2t}, x_{2t+h})) + \lambda(D_b(x_{2t}, Rx_{2t})) + \lambda(D_b(x_{2t+h}, Sx_{2t+h})) \\ +\lambda(D_b(x_{2t+h}, Rx_{2t})) + \lambda(D_b(x_{2t}, Sx_{2t+h})) \end{pmatrix}$$

$$= \psi\begin{pmatrix} \lambda(D_b(x_{2t}, x_{2t+h})) + \lambda(D_b(x_{2t}, x_{2t+1})) + \lambda(D_b(x_{2t+h}, x_{2t+1+h})) \\ +\lambda(D_b(x_{2t+h}, x_{2t+1})) + \lambda(D_b(x_{2t}, x_{2t+1+h})) \end{pmatrix}$$

$$\leq \psi\begin{pmatrix} \lambda\sum_{i=2t}^{2t+h-1}(D_b(x_i, x_{i+1})) + \lambda(D_b(x_{2t}, x_{2t+1})) + \lambda(D_b(x_{2t+h}, x_{2t+1+h})) \\ +\lambda\sum_{i=2t+1}^{2t+h-1}(D_b(x_i, x_{i+1})) + \lambda(D_b(x_{2t}, x_{2t+1+h})) \end{pmatrix}$$

$$\leq \psi\begin{pmatrix} \lambda(2+s)\sum_{i=2t}^{\infty}(D_b(x_i, x_{i+1})) + \lambda(D_b(x_{2t}, x_{2t+1})) \\ +\lambda(D_b(x_{2t+h}, x_{2t+1+h})) + \lambda s(D_b(x_{2t}, x_{2t+1})) \end{pmatrix}$$

$$Where, k = \lambda(2+s)$$

$$\leq \psi\begin{pmatrix} k\sum_{i=2t}^{\infty}(D_b(x_i, x_{i+1})) + \lambda(D_b(x_{2t}, x_{2t+1})) \\ +\lambda(D_b(x_{2t+h}, x_{2t+1+h})) + \lambda s(D_b(x_{2t}, x_{2t+1})) \end{pmatrix}$$

$$\leq \psi\begin{pmatrix} \frac{k^{2t+1}}{1-k}(D_b(x_0, x_1)) + \lambda(D_b(x_{2t}, x_{2t+1})) \\ +\lambda(D_b(x_{2t+h}, x_{2t+1+h})) + \lambda s(D_b(x_{2t}, x_{2t+1})) \end{pmatrix}$$

By permitting $n, m \to \infty$ in above inequalities and considering equation (9)

$$\lim_{n\to+\infty} \psi(D_b(x_n, x_m)) = 0$$

The properties of ψ implies that

$$\lim_{n\to+\infty} D_b(x_n, x_m) = 0 \tag{11}$$

Case 2:*n and m are both even integers. Applying the triangular inequality of the b-metric d_b, we have*

$$D_b(x_n, x_m) \leq s[D_b(x_n, x_{n+1}) + D_b(x_{n+1}, x_m)],\ for\ m \geq n$$

Letting $n \to \infty$ and in view of equation (10) and (11),we get

$$\lim_{n\to+\infty} \psi(D_b(x_n, x_m)) = 0.$$

Case 3: *n is an even integer and m is an odd integer. Applying the triangular inequality of the b-metric D_b, we have*

$$D_b(x_n, x_m) \leq s[D_b(x_n, x_{n+1}) + D_b(x_{n+1}, x_m)]$$

$$D_b(x_n, x_m) \leq s[D_b(x_n, x_{n+1}) + s[D_b(x_{n+1}, x_{m-1}) + D_b(x_{m-1}, x_m)]]\ ,\ for\ m \geq n$$

On permitting $m, n \to \infty$ and considering equation (10) and (11), we get

$$\lim_{n\to+\infty} \psi(D_b(x_n, x_m)) = 0.$$

Case 4: *n and m are both odd integers. Applying the triangular inequality of the b-metric D_b, we have*

$$D_b(x_n, x_m) \leq s[D_b(x_n, x_{m-1}) + D_b(x_{m-1}, x_m)],\ for\ m \geq n$$

On permitting $n \to \infty$ and in view of equation (10) and (11), we get

$$\lim_{n\to+\infty} \psi(D_b(x_n, x_m)) = 0.$$

Combining all cases with each other, we conclude that

$$\lim_{n\to+\infty} \psi(D_b(x_n, x_m)) = 0.$$

Thus, we conclude that (x_n) is a Cauchy sequence in X. The $\alpha,\ \beta$-completeness of the b-metric space (X, D_b) ensures that there is $x \in X$ such that $x_n \to x$. Using the $\alpha,\ \beta$-continuity of the mappings R and S, we deduce that $x_{2n+1} = Rx_{2n} \to Rx$ and $x_{2n+2} = Sx_{2n+1} \to Sx$. by uniqueness of limit, we obtain $Rx = Sx = x$. Thus, x is a fixed point of R.$\square$

Example 3.1. *Define* $D_b : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *by* $D_b(z,w) = |z-w|$ *and let* R, S *be two self-mappings on* $\mathbb{R}_0^+$ *define by* $Rz = \frac{z}{2}$ *and* $Sw = \frac{w}{4}$*. In addition, define the function* $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *by* $\psi^n \lambda x = \lambda^n \psi(x)$*, where* $\psi(x) = \frac{x}{1+x}$*. Furthermore, the functions* $\alpha, \beta : X \times X \to \mathbb{R}_0^+$ *define by*

$$\alpha(p,q) = \begin{cases} e^{p+q}, \ if \ p,q \in [0,1]; \\ 0, \ if \ p > 1 \text{ or } q > 1. \end{cases} \ . \text{ and } \beta(p,q) = \begin{cases} e^{p}, \ if \ p,q \in [0,1]; \\ 1, \ if \ p > 1 \text{ or } q > 1. \end{cases} \ .$$

Then:

1. ψ *is a perfect control function.*
2. *There exists* $u_0 \in X$ *such that*

   $\alpha(Ru_0, Ru_1) \geq \beta(Ru_0, Ru_1)$ *and* $\alpha(Ru_1, Ru_0)] \geq \beta(Ru_1, Ru_0)$*.*
3. (R, S) *is a pair of* (α, β)*-admissibility.*
4. R *and* S *are* α*,* β*-continuous.*
5. (X, D_b) *is an* α*,* β*-complete b-metric space.*
6. (R, S) *is an* (α, β, ψ)*-contraction.*

Proof. *It is an easy matter to see equations (1) to (3). To prove (4), let* (u_n) *be any sequence in* $\mathbb{R}_0^+$ *whenever* $u_n \to u \in \mathbb{R}_0^+$ *and* $\alpha(u_n, u_{n+1}) \geq \beta(u_n, u_{n+1}) \ \forall \ n \in N$*.*

Case 1: *If* $u_n = u$ *for all* n*, where* $u_n \in [0,1] \ \forall \ n \in \mathbb{N}$*. We conclude that* $Ru_n \to Ru$ *as* $n \to \infty$*.*

Case 2: *If* $u_n \neq u$*, for all n, we notice that* $u = 0$*. Hence,* $u_n \to 0$ *in* $([0,1], \ |.|)$*. Therefore,* $|\frac{u}{2}, 0| \to 0 = Ru$ *in* $(\mathbb{R}_0^+, D_b)$*; that is* R *is* α, β*-continuous.*

To prove (5)*, let* (u_n) *be a Cauchy sequence in* $(\mathbb{R}_0^+, D_b)$ *such that*

$$\alpha(u_n, u_{n+1}) \geq \beta(u_n, u_{n+1}).$$

Then, $u_n \in [0,1] \ \forall n \in \mathbb{N}$*. If there exists* $u \in [0,1]$ *such that* $u_n = u$ *for all* n*, then,* $u_n \to u$ *as* $n \to +\infty$*. Now, suppose the elements of* (u_n) *are distinct. Give* $\epsilon > 0$*, since* (u_n) *is a Cauchy sequence in* $(\mathbb{R}_0^+, D_b)$*, then there exists* $n_0 \in \mathbb{N}$ *such that* $|u_n, u_m| < \epsilon \ \forall \ \ m > n \geq n_0$*. Therefore,* $|u_n, 0| < 0 \ \forall \ n \geq n_0$*. So,* $u_n \to 0$ *in* $(\mathbb{R}_0^+, D_b)$*. Thus,* $(\mathbb{R}_0^+, D_b)$ *is an* α, β*-complete b-metric space.*

To prove (6)*, let* $z, w \in X$ *be such that* $\alpha(z,w) \geq \beta(z,w)$*. Then,* $z, w \in [0,1]$*.*

So,

$$\begin{aligned}\psi(D_b(Rz,Sw)) &= \psi(D_b(\tfrac{z}{2},\tfrac{w}{4}))\\ &= \psi(|\tfrac{z}{2},\tfrac{w}{4}|)\\ &= \frac{|\frac{z}{2},\frac{w}{4}|}{1+|\frac{z}{2},\frac{w}{4}|}\\ &= \frac{\frac{1}{2}|z,\frac{w}{2}|}{1+\frac{1}{2}|z,\frac{w}{2}|}\\ &= \frac{|z,\frac{w}{2}|}{2+|z,\frac{w}{2}|}\\ &\leq \frac{1}{6}(\frac{|z,w|}{1+|z,w|})\\ &\leq \frac{1}{6}\psi(D_b(z,w))\end{aligned}$$

$$\begin{aligned}\psi(D_b(Rz,Sw)) &\leq \tfrac{1}{6}\psi(D_b(z,w))+\tfrac{1}{6}\psi(D_b(z,Rz))+\tfrac{1}{6}\psi(D_b(w,Sw))+\\ &\tfrac{1}{6}\psi(D_b(w,Rz))+\tfrac{1}{6}\psi(D_b(z,Sw)),\ (\because\ \lambda=\tfrac{1}{6})\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}\psi(D_b(Sz,Rw)) &\leq \tfrac{1}{6}\psi(D_b(z,w))+\tfrac{1}{6}\psi(D_b(z,Sz))+\tfrac{1}{6}\psi(D_b(w,Rw))+\\ &\tfrac{1}{6}\psi(D_b(w,Sz))+\tfrac{1}{6}\psi(D_b(z,Rw)),\ (\because\ \lambda=\tfrac{1}{6})\end{aligned}$$

Hence, R and S satisfy definition 3.2. Therefore, R and S satisfy all the condition of theorem. Therefore, R and S have a common fixed point.

4 Conclusions

In Theorem 3.1 we have formulated a new contractive conditions to modify and extend some common fixed point theorem for a pair of self-mappings (R, S) in b-metric space via (α, β, ψ)-admissibility type. The existence and uniqueness of the result is presented in this article. We have also given some example which satisfies the condition of our main result. Our result may be the vision for other authors to extend and improve several results in such spaces and applications to other related areas.

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