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Common fixed point theorem for weakly compatible mappings in S_m metric space

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Abstract

In the present paper, at first, we study the structure of the newly S_m- metric space, which is a combination of S-metric space and multiplicative metric space. We have proved a common fixed point theorem for four self-maps in S_m metric space with a new contraction condition by applying the concepts of weakly compatible mappings, semi-compatible mappings, and reciprocally continuous mappings. Further, we also provide some examples to support our results.

Keywords: Multiplicative metric space, S-metric space,S_m-metric space, weakly compatible mappings, reciprocally continuous mappings, and semi-compatible mappings.
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1 Introduction

The notion of multiplicative metric space (MMS) was first developed by Bashirove [1]. Following that, several theorems came to light in this area of MMS [2] ,[3] and [4]. On the other side, Sedghi.S et al.[5] presented a new structure to S-metric space which modified D-metric and G-metric spaces, and then several fixed point theorems [6] and [7] were obtained. Pant et al. [8] generalized the notion of reciprocally continuous mapping which is weaker than continuous and compatible mappings. Recently, Mukesh Kumar Jain [9] introduced a more general form of semi-compatible mappings and proved many fixed point theorems in metric space.
In this article, we use a new generalized metric space referred to as S_m -metric space, which is a combination of both MMS and S -metric space. Using this concept, we establish a common fixed point theorem by applying weakly compatible mappings(WCM), reciprocally continuous mappings, and semi-compatible mappings. Furthermore, some examples are also discussed to support our conclusions.

2 Preliminaries:

Now we give some definitions and examples which are used in this theorem.

Definition 2.1. *[1] "Let χ be a non-empty set and $\delta : \chi^2 \to \mathbb{R}^+$ be a multiplicative metric space (MMS) satisfying the properties :*

(i) $\delta(\psi,\phi) \geq 1$ *and* $\delta(\psi,\phi) = 1 \iff \psi = \phi$

(ii) $\delta(\psi,\phi) = \delta(\phi,\psi)$

(iii) $\delta(\psi,\phi) \leq \delta(\psi,\sigma)\delta(\sigma,\phi), \forall \psi,\phi,\sigma \in \chi$."

Definition 2.2. *[5] " Let χ be a non-empty set defined $S : \chi^3 \to [0,\infty)$ satisfying:*

(i) $S(\psi,\phi,\sigma) \geq 0$

(ii) $S(\psi,\phi,\sigma) = 0 \iff \psi = \phi = \sigma$

(iii) $S(\psi,\phi,\sigma) \leq S(\psi,\psi,\rho) + S(\phi,\phi,\rho) + S(\sigma,\sigma,\rho), \forall \psi,\phi,\sigma,\rho \in \chi$.

A mapping S together with χ, (χ, S) is called a S-metric space."

Definition 2.3. *[10] " Let χ be a non-empty set .A function $S_m : \chi^3 \to \mathbb{R}^+$ satisfying the conditions :*

(i) $S_m(\psi, \phi, \sigma) \geq 1$

(ii) $S_m(\psi, \phi, \sigma) = 1 \iff \psi = \phi = \sigma$

(iii) $S_m(\psi, \phi, \sigma) \leq S_m(\psi, \psi, \rho) S_m(\phi, \phi, \rho) S_m(\sigma, \sigma, \rho), \forall \psi, \phi, \sigma, \rho \in \chi$.

The pair (χ, S_m) *is called as* S_m*-metric space".*

Definition 2.4. *[10] " Let* (χ, S_m) *be a* S_m*-metric space, a sequence* $\{\psi_\theta\} \in \chi$ *is said to be*

(i) cauchy sequence $\iff S_m(\psi_\theta, \psi_\theta, \psi_l) \to 1,$ *for all* $\theta, l \to \infty$;

(ii) convergent $\iff \exists \psi \in \chi$ *such that* $S_m(\psi_\theta, \psi_\theta, \psi) \to 1$ *as* $\theta \to \infty$;

(iii) is complete if every cauchy sequence is convergent."

Definition 2.5. *[11] " Two self-maps* M *and* K *of a* S_m *metric space are said to be*

(i) ***Compatible:*** *if*

$$\lim_{\theta \to \infty} S_m(MK\psi_\theta, MK\psi_\theta, KM\psi_\theta) = 1,$$

whenever there exist a sequence $\{\psi_\theta\} \in \chi$ *such that*

$$\lim_{\theta \to \infty} S_m(M\psi_\theta, K\psi_\theta, \omega) = 1 \textit{ for some } \omega \in \chi.$$

(ii) ***Weakly- compatible mappings:*** *if they commute at their coincidence points,*

$$i.e. \omega \in \chi, S_m(M\omega, M\omega, K\omega) = 1, \implies S_m(MK\omega . MK\omega, KM\omega) = 1."$$

Definition 2.6. *[9] "Two self maps* M *and* K *of* S_m*-metric space are said to be* ***Semi- compatible:*** *if*

$$\lim_{\theta \to \infty} S_m(MK\psi_\theta, MK\psi_\theta, K\omega) = 1$$

whenever there exists a sequence $\{\psi_\theta\} \in X$ *such that*

$$\lim_{\theta \to \infty} S_m(M\psi_\theta, K\psi_\theta, \omega) = 1 \textit{ for all } \omega \in \chi."$$

Now we present an example in which semi-compatible is weaker than compatible.

Example 2.6.1
Consider $\chi = [0, \infty)$ with $S_m(\psi, \phi, \sigma) = e^{|\psi-\phi|+|\phi-\sigma|+|\sigma-\psi|}$, for every $\psi, \phi, \sigma \in \chi$. Define two self maps M and K as

$$M(\psi) = \begin{cases} \frac{cos^2(\pi\psi)+1}{2} & \text{if } 0 < \psi \leq \frac{1}{2}; \\ sin(\pi\psi) & \text{if } \frac{1}{2} < \psi \leq 3. \end{cases}$$

and

$$K(\psi) = \begin{cases} \frac{2sin(\pi\psi)-1}{2} & \text{if } 0 < \psi \leq \frac{1}{2}; \\ 1 - sin(\pi\psi) & \text{if } \frac{1}{2} < \psi \leq 3. \end{cases}$$

Consider a sequence $\{\psi_\theta\}$ as $\psi_\theta = \{\frac{\pi}{2} - \frac{1}{\theta}\}$ for $\theta \geq 0$.
Then

$$\lim_{\theta\to\infty} M(\psi_\theta) = \lim_{\theta\to\infty} M(\frac{1}{2}-\frac{1}{\theta}) = \lim_{\theta\to\infty} \frac{cos^2\pi(\frac{1}{2}-\frac{1}{\theta})+1}{2} = \lim_{\theta\to\infty} \frac{sin^2(\frac{\pi}{\theta})+1}{2} = \frac{1}{2}$$

and

$$\lim_{\theta\to\infty} K(\psi_\theta) = \lim_{\theta\to\infty} K(\frac{1}{2}-\frac{1}{\theta}) = \lim_{\theta\to\infty} \frac{2sin\pi(\frac{1}{2}-\frac{1}{\theta})-1}{2} = \lim_{\theta\to\infty} \frac{2cos(\frac{\pi}{\theta})-1}{2} = \frac{1}{2}.$$

$$\text{Therefore } \lim_{\theta\to\infty} M\psi_\theta = \lim_{\theta\to\infty} K\psi_\theta = \frac{1}{2} = \omega \text{ (say)}.$$

Now

$$\lim_{\theta\to\infty} MK(\psi_\theta) = \lim_{\theta\to\infty} M(\frac{2cos\frac{\pi}{\theta}-1}{2}) = \lim_{\theta\to\infty} \frac{cos^2\pi(\frac{2cos\frac{\pi}{\theta}-1}{2})+1}{2} = \frac{cos^2\frac{\pi}{2}+1}{2} = \frac{1}{2}$$

and

$$\lim_{\theta\to\infty} KM(\psi_\theta) = \lim_{\theta\to\infty} K(\frac{sin^2\frac{\pi}{\theta}+1}{2}) = \lim_{\theta\to\infty} [1 - sin\pi(\frac{sin^2\frac{\pi}{\theta}+1}{2})] = 0.$$

$$\therefore \lim_{\theta\to\infty} S_m(MK\psi_\theta, MK\psi_\theta, KM\psi_\theta) \neq 0.$$

This implies these two self-maps M and K are not compatible.
But $K(\omega) = K(\frac{1}{2}) = \frac{1}{2}$.

$$\text{Therefore } \lim_{\theta\to\infty} S_m(MK\psi_\theta, MK\psi_\theta, K\omega) = \lim_{\theta\to\infty} S_m(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 1.$$

Hence these two self maps M and K are semi-compatible but not compatible.

Definition 2.7. *[8] "Two self-maps $M,\ K$ of S_m-metric space are said to be* ***reciprocally continuous*** *if*

$$\lim_{\theta\to\infty} S_m(MK\psi_\theta, MK\psi_\theta, M\omega) = 1 \text{ and } \lim_{\theta\to\infty} S_m(KM\psi_\theta, KM\psi_\theta, K\omega) = 1,$$

whenever there exist a sequence $\{\psi_\theta\} \in \chi$ such that

$$\lim_{\theta\to\infty} S_m(M\psi_\theta, K\psi_\theta, \omega) = 1 \text{ some } \omega \in \chi."$$

Now we present an example in which satisfies reciprocally continuous is weaker but not compatible.

Example 2.7.1 Consider $\chi = (0, \infty)$ with $S_m(\psi, \phi, \sigma) = e^{|\psi-\phi|+|\phi-\sigma|+|\sigma-\psi|}$, for every $\psi, \phi, \sigma \in \chi$. Define two self maps M and K as

$$M(\psi) = \begin{cases} \psi^2 + 2 & \text{if } 0 < \psi \le 1; \\ 4 - \psi & \text{if } 1 < \psi \le 3. \end{cases}$$

and

$$K(\psi) = \begin{cases} 1 - 2\psi & \text{if } 0 < \psi \le 1; \\ \psi - 2 & \text{if } 1 < \psi \le 3. \end{cases}$$

Consider a sequence $\{\psi_\theta\}$ as $\psi_\theta = \{3 - \frac{1}{\theta}\},$ for $\theta \ge 0$.
Now

$$\lim_{\theta\to\infty} M(\psi_\theta) = \lim_{\theta\to\infty} [4 - (3 - \frac{1}{\theta})] = 1 \text{ and } \lim_{\theta\to\infty} K(\psi_\theta) = \lim_{\theta\to\infty} [(3 + \frac{1}{\theta}) - 2] = 1$$

$$\therefore \lim_{\theta\to\infty} M\psi_\theta = \lim_{\theta\to\infty} K\psi_\theta = 1 = \omega_1 \neq \phi.$$

Also

$$\lim_{\theta\to\infty} MK(\psi_\theta) = \lim_{\theta\to\infty} M[(3 - \frac{1}{\theta}) - 2] = \lim_{\theta\to\infty} M(1 - \frac{1}{\theta}) = 3$$

and

$$\lim_{\theta\to\infty} KM(\psi_\theta) = \lim_{\theta\to\infty} K(4 - (3 - \frac{1}{\theta}) = \lim_{\theta\to\infty} K(1 + \frac{1}{\theta}) = -1.$$

$$\therefore \lim_{\theta\to\infty} S_m(MK\psi_\theta, MK\psi_\theta, KM\psi_\theta) = S_m(3, 3, -1) \neq 1.$$

This gives the self maps M and K are not compatible in S_m- metric space.
Moreover,$M(\omega_1) = 3$ and $K(\omega_1) = -1$.
Which gives

$$\lim_{\theta\to\infty} S_m(MK\psi_\theta, MK\psi_\theta, M\omega_1) = S_m(3, 3, 3) = 1,$$

and

$$\lim_{\theta\to\infty} S_m(KM\psi_\theta, KM\psi_\theta, K\omega_1) = S_m(-1,-1,-1) = 1.$$

This implies the self-maps M and K are reciprocally continuous but not compatible in S_m metric space.

Now we proceed to the main theorem.

3 Main Theorem

Theorem 3.1. *Let M, H, K, and J be self-mapping of a complete S_m-metric space satisfying the following*

(3.1.1) $M(\chi) \subseteq J(\chi)$ *and* $H(\chi) \subseteq K(\chi)$

(3.1.2)

$$S_m(M\psi, M\psi, H\phi) \leq \Big\{ max[S_m(M\psi, M\psi, K\psi)S_m(H\phi, H\phi, J\phi), \\ S_m(M\psi, M\psi, J\phi)S_m(K\psi, K\psi, H\phi), \\ S_m(M\psi, M\psi, J\phi)S_m(H\phi, H\phi, J\phi), \\ S_m(M\psi, M\psi, K\psi)S_m(H\phi, H\phi, K\psi)] \Big\}^\lambda \\ \textit{where } \lambda \in (0, \frac{1}{2})$$

(3.1.3) the pair M and K are reciprocally continuous and semi-compatible,

(3.1.4) the pair H and J are weakly compatible.

Then the self-maps M, H, K, and J have a unique common fixed point in χ.

Proof:
Let there is a point $\psi_0 \in \chi$, and the sequence $\{\psi_\theta\}$ be defined as $M\psi_0 = J\psi_1 = \phi_0$. For this point ψ_1 then there exists $\psi_2 \in \chi$ such that $H\psi_1 = K\psi_2 = \phi_1$. In general, by induction choose $\psi_{\theta+1}$, construct a sequence $\{\phi_\theta\} \in \chi$ such that

$$\phi_{2\theta} = M\psi_{2\theta} = J\psi_{2\theta+1} \textit{ and } \phi_{2\theta+1} = H\psi_{2\theta+1} = K\psi_{2\theta+2}, \textit{ for } \theta \geq 0.$$

On putting $\psi = \psi_{2\theta}$ and $\phi = \phi_{2\theta+1}$ in (3.1.2) we get.

$$S_m(\phi_{2\theta},\phi_{2\theta},\phi_{2\theta+1}) = S_m(M\psi_{2\theta}, M\psi_{2\theta}, H\psi_{2\theta+1})$$
$$\leq \textit{ max} \Big\{ S_m(M\psi_{2\theta}, M\psi_{2\theta}, \theta\psi_{2\theta})S_m(H\psi_{2\theta+1}, H\psi_{2\theta+1}, J\psi_{2\theta+1}),$$
$$S_m(M\psi_{2\theta}, M\psi_{2\theta}, J\psi_{2\theta+1})S_m(H\psi_{2\theta+1}, H\psi_{2\theta+1}, \theta\psi_{2\theta}),$$
$$S_m(M\psi_{2\theta}, M\psi_{2\theta}, J\psi_{2\theta+1})S_m(H\psi_{2\theta+1}, H\psi_{2\theta+1}, J\psi_{2\theta+1}),$$
$$S_m(M\psi_{2\theta}, M\psi_{2\theta}, K\psi_{2\theta})S_m(H\psi_{2\theta+1}, H\psi_{2\theta+1}, K\psi_{2\theta}) \Big\}^{\lambda}$$

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq \textit{ max} \Big\{ S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta-1})S_m(\phi_{2\theta+1}, \phi_{2\theta+1}, \phi_{2\theta}),$$
$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta})S_m(\phi_{2\theta+1}, \phi_{2\theta+1}, \phi_{2\theta-1}),$$
$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta})S_m(\phi_{2\theta+1}, \phi_{2\theta+1}, \phi_{2\theta}),$$
$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta-1})S_m(\phi_{2\theta+1}, \phi_{2\theta+1}, \phi_{2\theta-1}) \Big\}^{\lambda}$$

this implies that

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq S_m(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta+1})^{\lambda}.$$

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq \{S_m(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta})S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1})\}^{\lambda}.$$

$$S_m^{1-\lambda}(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq S_m^{\lambda}(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta}).$$

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq S_m^{\frac{\lambda}{1-\lambda}}(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta}).$$

$$S_m(\phi_{2\theta}, \phi_{2\theta}, \phi_{2\theta+1}) \leq S_m^{p}(\phi_{2\theta-1}, \phi_{2\theta-1}, \phi_{2\theta}). \textit{ where } p = \frac{\lambda}{1-\lambda}.$$

Now this gives

$$S_m(\phi_\theta, \phi_\theta, \phi_{\theta+1}) \leq S_m^{p}(\phi_{\theta-1}, \phi_{\theta-1}, \phi_\theta) \leq S_m^{p^2}(\phi_{\theta-2}, \phi_{\theta-2}, \phi_{\theta-1}) \leq \cdots S_m^{p^n}(\phi_0, \phi_0, \phi_n).$$

By using triangular inequality

$$S_m(\phi_\theta, \phi_\theta, \phi_n) \leq S_m^{p^\theta}(\phi_0, \phi_0, \phi_l) \leq S_m^{p^{\theta+1}}(\phi_0, \phi_0, \phi_n) \leq \cdots S_m^{p^{n-1}}(\phi_0, \phi_0, \phi_n)$$

$$S_m(\phi_\theta, \phi_\theta, \phi_n) \le S_m^{\frac{p^\theta}{1-p}}(\phi_0, \phi_0, \phi_l) \textit{ for all } \theta \ge 1.$$

Hence $\{\phi_\theta\}$ is a cauchy sequence in S_m-metric space.
Since the self-maps, M and K are weakly reciprocally continuous.

$$\lim_{\theta\to\infty} S_m(MK\psi_\theta, MK\psi_\theta, M\omega) = 1 \textbf{\textit{ or }} \lim_{\theta\to\infty} S_m(KM\psi_\theta, KM\psi_\theta, \theta\omega) = 1. \quad (1)$$

Also, the pair (M, K) is semi compatible, we have

$$\lim_{\theta\to\infty} S_m(MK\psi_\theta, MK\psi_\theta, K\omega) = 1. \quad (2)$$

From (1) *and* (2) *we get*

$$S_m(M\omega, M\omega, K\omega) = 1. \quad (3)$$

Since $M(\chi) \subseteq J(\chi)$ which gives then there exists $\nu \in \chi$ such that $J\nu = M\psi_\theta$, since $M\psi_\theta \to \omega$ as $\theta \to \infty$. Which implies

$$S_m(J\nu, J\nu, \omega) = 1. \quad (4)$$

Now, we have to prove $S_m(J\nu, H\nu, \omega) = 1$.
Substitute $\psi = \psi_\theta$ and $\phi = \nu$ in (3.1.2) *we have*

$$\begin{aligned} S_m(M\psi_\theta, M\psi_\theta, H\nu) \le \Big\{ & max[S_m(M\psi_\theta, M\psi_\theta, K\psi_\theta)S_m(H\nu, H\nu, J\nu), \\ & S_m(M\psi_\theta, M\psi_\theta, J\nu)S_m(K\psi_1, K\psi_1, H\nu), \\ & S_m(M\psi_\theta, M\psi_\theta, J\nu)S_m(H\nu, H\nu, J\nu), \\ & S_m(M\psi_\theta, M\psi_\theta, K\psi_\theta)S_m(H\nu, H\nu, K\psi_\theta)] \Big\}^\lambda \end{aligned}$$

$$\begin{aligned} & S_m(\omega, \omega, H\nu) \le \\ & \Big\{ max[S_m(\omega, \omega, \omega)S_m(H\nu, H\nu, \omega), S_m(\omega, \omega, \omega)S_m(\omega, \omega, H\nu), \\ & S_m(\omega, \omega, \omega)S_m(H\nu, H\nu, \omega), S_m(\omega, \omega, \omega)S_m(H\nu, H\nu, \omega)] \Big\}^\lambda \end{aligned}$$

$$S_m(\omega, \omega, H\nu) \le \{(S_m(\omega, \omega, H\nu)\}^\lambda$$

$$S_m^{(1-\lambda)}(\omega, \omega, H\nu) \le 1 \implies S_m(H\nu, H\nu, \omega) = 1.$$

$$\therefore S_m(J\nu, H\nu, \omega) = 1.$$

Since the pair $(H.J)$ *is WCM and* ν *is a coincidence point then* $HJ\nu = JH\nu$

$$S_m(H\omega, H\omega, J\omega) = 1. \tag{5}$$

Substitute $\psi = \psi_\theta$ *and* $\phi = \omega$ *in* $(3.1.2)$
we have

$$\begin{aligned} & S_m(M\psi_\theta, M\psi_\theta, H\omega) \leq \\ & \Big\{ max[S_m(M\psi_\theta, M\psi_\theta, K\psi_\theta)S_m(H\omega, H\omega, J\omega), \\ & S_m(M\psi_\theta, M\psi_\theta, J\omega)S_m(K\psi_1, K\psi_1, H\omega), \\ & S_m(M\psi_\theta, M\psi_\theta, J\omega)S_m(H\omega, H\omega, J\omega), \\ & S_m(M\psi_\theta, M\psi_\theta, K\psi_\theta)S_m(H\omega, H\omega, K\psi_\theta)] \Big\}^\lambda \end{aligned}$$

also

$$\begin{aligned} & S_m(H\omega, \omega, \omega) \leq \\ & \Big\{ max[S_m(\omega, \omega, \omega)S_m(H\omega, H\omega, \omega), S_m(\omega, \omega, \omega)S_m(\omega, \omega, H\omega), \\ & S_m(\omega, \omega, \omega)S_m(H\omega, H\omega, \omega), S_m(\omega, \omega, \omega)S_m(H\omega, H\omega, \omega)] \Big\}^\lambda \end{aligned}$$

and this gives

$$S_m(H\omega, \omega, \omega) \leq S_m(H\omega, \omega, \omega)^\lambda$$

$$S_m^{(1-\lambda)}(H\omega, \omega, \omega) \leq 1 \implies H\omega = \omega$$

$$\therefore S_m(H\omega, J\omega, \omega) = 1. \tag{6}$$

Replace $\psi = \omega$ *and* $\phi = \nu$ *in* $(3.1.2)$ *then we have*

$$\begin{aligned} S_m(M\omega, M\omega, H\nu) \leq & \Big\{ max[S_m(M\omega, M\omega, K\omega)S_m(J\nu, H\nu, H\nu), \\ & S_m(M\omega, M\omega, J\nu)S_m(K\omega, K\omega, H\nu), \\ & S_m(M\omega, M\omega, J\nu)S_m(J\nu, J\nu, H\nu), \\ & S_m(M\omega, M\omega, K\omega)S_m(H\nu, H\nu, K\omega)] \Big\}^\lambda \end{aligned}$$

$$S_m(M\omega, M\omega, \omega) \leq \Big\{ max[S_m(M\omega, M\omega, M\omega)S_m(\omega,\omega,\omega), S_m(M\omega, M\omega, \omega)S_m(M\omega, M\omega, \omega), S_m(M\omega, M\omega, \omega)S_m(\omega,\omega,\omega), S_m(M\omega, M\omega, M\omega)S_m(\omega,\omega, M\omega)] \Big\}^{\lambda}$$

$$S_m(M\omega, M\omega, \omega) \leq \{S_m(M\omega, M\omega, \omega)\}^{\lambda}$$

$$S_m^{(1-\lambda)}(M\omega, M\omega, \omega) \leq 1 \implies M\omega = \omega$$

$$\therefore S_m(M\omega, J\omega, \omega) = 1. \tag{7}$$

From (6) *and* (7) *we get*

$$M\omega = J\omega = H\omega = K\omega = \omega. \tag{8}$$

Therefore "ω" *is a common fixed point of **M, H, K, and J.***

Uniqueness

Let ρ *be one more fixed point, we assume that* $\rho \neq \omega$ *then we have*

$$M\rho = K\rho = H\rho = J\rho = \rho.$$

In the condition (3.1.2) put $\psi = \omega$ *and* $\phi = \rho$ *we get*

$$S_m(M\omega, M\omega, H\rho) \leq \Big\{ max[S_m(M\omega, M\omega, K\omega)S_m(H\rho, H\rho, J\rho), S_m(M\omega, M\omega, J\rho)S_m(K\omega, K\omega, H\rho), S_m(M\omega, M\omega, J\rho)S_m(H\rho, H\rho, J\rho), S_m(M\omega, M\omega, K\omega)S_m(H\rho, H\rho, K\omega)] \Big\}^{\lambda}$$

$$S_m(\omega,\omega,\rho) \leq \Big\{ max[S_m(\omega,\omega,\omega)S_m(\rho,\rho,\rho), S_m(\omega,\omega,\rho)S_m(\omega,\omega,\rho), S_m(\omega,\omega,\rho)S_m(\rho,\rho,\rho), S_m(\omega,\omega,K\omega)S_m(\rho,\rho,\omega)] \Big\}^{\lambda}$$

$$S_m(\omega,\omega,\rho) \leq \Big\{ S_m(\omega,\omega,\rho) \Big\}^{\lambda}$$

this implies that $S_m(\omega,\omega,\rho) = 1 \implies \omega = \rho$.

This shows that "ω" *is the unique common fixed point of M.H.J and K.*

Now, the following example substantiates our theorem.

Example 3.2

Suppose $\chi = (0,1)$, S_m- metric space by $S_m(\psi,\phi,\sigma) = e^{|\psi-\phi|+|\phi-\sigma|+|\sigma-\psi|}$,

when $\psi,\phi,\sigma \in \chi$. Define M ,K ,H J:$\chi X \chi \to \chi$ as follows

$$M(\psi) = \begin{cases} \frac{2-\psi}{5} & \text{if } 0 < \psi \leq \frac{1}{3}; \\ \psi & \text{if } \frac{1}{3} < \psi < 1. \end{cases}$$

$$K(\psi) = \begin{cases} 1-2\psi & \text{if } 0 < \psi \leq \frac{1}{3}; \\ \frac{1+\psi}{2} & \text{if } \frac{1}{3} < \psi < 1. \end{cases}$$

$$H(\psi) = \begin{cases} 3\psi^2 - 3\psi + 1 & \text{if } 0 < \psi \leq \frac{1}{3}; \\ \frac{2+\psi}{7} & \text{if } \frac{1}{3} < \psi < 1. \end{cases}$$

$$J(\psi) = \begin{cases} 1-6\psi^2 & \text{if } 0 < \psi \leq \frac{1}{3}; \\ 1-\psi & \text{if } \frac{1}{3} < \psi < 1. \end{cases}$$

Then $M(\chi) = (\frac{1}{3}, 1] \subseteq J(\chi) = (0,1]$ and $H(\chi) = (\frac{1}{3}, 1] \subseteq K(\chi) = (\frac{1}{3}, 1]$.

Therefore the condition (3.1.1) holds.

Consider a sequence $\{\psi_\theta\}$ as $\psi_\theta = \{\frac{1}{3} - \frac{1}{\theta}\}$ as $\theta \geq 0$.

$$\text{Then } \lim_{\theta\to\infty} M(\psi_\theta) = \lim_{\theta\to\infty} M(\frac{1}{3} - \frac{1}{\theta}) = \lim_{\theta\to\infty} \frac{2 - (\frac{1}{3} - \frac{1}{\theta})}{5} = \frac{1}{3}$$

and

$$\lim_{\theta\to\infty} K(\psi\theta) = \lim_{\theta\to\infty} K(\frac{1}{3} - \frac{1}{\theta}) = \lim_{\theta\to\infty} [1 - 2(\frac{1}{3} - \frac{1}{\theta})] = \frac{1}{3}.$$

$$\text{Therefore } \lim_{\theta\to\infty} M(\psi_\theta) = \lim_{\theta\to\infty} K(\psi_\theta) = \frac{1}{3} = \omega_1.$$

Further

$$\lim_{\theta\to\infty} H(\psi_\theta) = \lim_{\theta\to\infty} H(\frac{1}{3} - \frac{1}{\theta}) = \lim_{\theta\to\infty} [3(\frac{1}{3} - \frac{1}{\theta})^2 - 3(\frac{1}{3} - \frac{1}{\theta}) + 1] = \frac{1}{3}$$

and

$$\lim_{\theta\to\infty} J(\psi_\theta) = \lim_{\theta\to\infty} J(\frac{1}{3} - \frac{1}{\theta}) = \lim_{\theta\to\infty} [1 - 6(\frac{1}{3} - \frac{1}{\theta})^2] = \frac{1}{3}.$$

$$\text{Therefore } \lim_{\theta\to\infty} H(\psi_\theta) = \lim_{\theta\to\infty} J(\psi_\theta) = \frac{1}{3} = \omega_1.$$

Moreover

$$\lim_{\theta\to\infty} MK(\psi_\theta) = \lim_{\theta\to\infty} M[1-(\frac{2}{3}-\frac{2}{\theta})] = \lim_{\theta\to\infty} M(\frac{1}{3}+\frac{2}{\theta}) = \frac{1}{3}$$

and

$$\lim_{\theta\to\infty} KM(\psi_\theta) = \lim_{\theta\to\infty} K(\frac{1}{3}+\frac{1}{5\theta}) = \lim_{\theta\to\infty} \frac{1+2(\frac{1}{3}+\frac{1}{5\theta}}{2}) = \frac{2}{3}.$$

$$\therefore \lim_{\theta\to\infty} S_m(MK\psi_\theta, MK\psi_\theta, KM\psi_\theta) = S_m(\frac{1}{3},\frac{1}{3},\frac{2}{3}) \neq 1$$

which implies that the pair (M, K) is not compatible.
Furthermore

$$\lim_{\theta\to\infty} HJ(\psi_\theta) = \lim_{\theta\to\infty} H(\frac{1}{3}+\frac{4}{\theta}-\frac{1}{\theta^2}) = \lim_{\theta\to\infty} (\frac{2+(\frac{1}{3}+\frac{4}{\theta}-\frac{1}{\theta^2}}{7}) = \frac{1}{3}$$

and

$$\lim_{\theta\to\infty} JH(\psi_\theta) = \lim_{\theta\to\infty} J(\frac{1}{3}+\frac{4}{\theta}-\frac{1}{\theta^2}) = \lim_{\theta\to\infty} [1-(\frac{1}{3}+\frac{4}{\theta}-\frac{1}{\theta^2})] = \frac{2}{3}$$

Therefore $\lim_{\theta\to\infty} S_m(HJ\psi_\theta, HJ\psi_\theta, JH\psi_\theta) = S_m(\frac{1}{3},\frac{1}{3},\frac{2}{3}) \neq 1.$

Which shows that the pair(H,J) is not compatible .

Also $M(\frac{1}{3}) = \frac{1}{3}, K(\frac{1}{3}) = \frac{1}{3}$.

This implies $\lim_{\theta\to\infty} S_m(MK\psi_\theta, MK\psi_\theta, M\omega_1) = S_m(\frac{1}{3},\frac{1}{3},\frac{1}{3}) = 1$

and $\lim_{\theta\to\infty} S_m(KM\psi_\theta, KM\psi_\theta, K\omega_1) = S_m(\frac{1}{3},\frac{1}{3},\frac{1}{3}) = 1.$

This shows that the pair (M, K) is reciprocally continuous in S_m metric space.

Also $\lim_{\theta\to\infty} S_m(MK\psi_\theta, MK\psi_\theta, K\omega_1) = S_m(\frac{1}{3},\frac{1}{3},\frac{1}{3}) = 1.$

This shows that the pair (M, K) is semi-compatible in S_m metric space.

Hence the inequality (3.1.3)holds.

Further

$$S_m(H(\frac{1}{3}), J(\frac{1}{3}), \frac{1}{3}) = 1 \text{ and } S_m(HJ(\frac{1}{3}), JH(\frac{1}{3}), \frac{1}{3}) = 1.$$

This implies that $S_m(HJ(\frac{1}{3}), HJ(\frac{1}{3}), JH(\frac{1}{3})) = S_m(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1$.
Which indicates that the pair (H, J) is weakly compatible.

Now, we prove the condition (3.1.2) in various cases

CASE-I
Let $\psi, \phi \in [0, \frac{1}{2}]$,while we have $S_m(\psi, \phi, \sigma) = e^{|\psi-\sigma|+|\phi-\sigma|}$.
Take $\psi = \frac{1}{4}$ and $\phi = \frac{1}{5}$ then $M(\frac{1}{4}) = \frac{7}{20}, K(\frac{1}{4}) = \frac{1}{2}, H(\frac{1}{5}) = \frac{13}{25}$ and $J(\frac{1}{5}) = \frac{19}{25}$
substitute the above values in **(3.1.2)**

$$S_m(\frac{7}{20}, \frac{7}{20}, \frac{13}{25}) \leq$$
$$\Big\{max[S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{13}{25}, \frac{13}{25}, \frac{19}{25}), S_m(\frac{7}{20}, \frac{7}{20}, \frac{19}{25})S_m(\frac{13}{25}, \frac{13}{25}, \frac{1}{2}),$$
$$S_m(\frac{7}{20}, \frac{7}{20}, \frac{19}{25})S_m(\frac{13}{25}, \frac{13}{25}, \frac{19}{25}), S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{13}{25}, \frac{13}{25}, \frac{1}{2})]\Big\}^{\lambda}$$

$$wehavee^{0.34} \leq \left\{ \text{max}[e^{0.3}e^{0.48}, e^{0.82}e^{0.34}, e^{0.3}e^{0.04}, e^{0.82}e^{0.48}] \right\}^{\lambda}$$

$$e^{0.34} \leq \{ \text{ max}[e^{0.78}, e^{1.16}, e^{0.0.34}, e^{1.3}]\}^{\lambda} \implies e^{0.34} \leq e^{1.16\lambda}$$

which gives $\lambda = 0.2$ where $\lambda \in (0, \frac{1}{3})$.

CASE-II
Let $\psi, \phi \in (\frac{1}{2}, 1]$, then $S_m(\psi, \phi, \sigma) = e^{|\psi-\sigma|+|\phi-\sigma|}$.
Take $\psi = \frac{1}{2}$ and $\phi = \frac{1}{2}$ then $M(\frac{1}{2}) = \frac{1}{2}, K(\frac{1}{2}) = \frac{3}{4}, H(\frac{1}{2}) = \frac{5}{14}$ and $J(\frac{1}{2}) = \frac{1}{2}$
substitute the above values in **(3.1.2)**

$$S_m(\frac{1}{2}, \frac{1}{2}, \frac{5}{14}) \leq$$
$$\Big\{max[S_m(\frac{1}{2}, \frac{1}{2}, \frac{3}{4})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}), S_m(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})S_m(\frac{5}{14}., \frac{5}{14}, \frac{3}{4}),$$
$$S_m(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}), S_m(\frac{1}{2}, \frac{1}{2}, \frac{3}{4})S_m(\frac{5}{14}, \frac{5}{14}, \frac{3}{4})]\Big\}^{\lambda}$$

which implies that

$$e^{0.285} \leq \left\{ \text{max}[e^{0.5}e^{0.285}, e^{0.0}e^{0.786}, e^{0.0}e^{0.28}, e^{0.5}e^{0.786}] \right\}^{\lambda}$$

$$e^{0.285} \leq \{ \text{ max}[e^{0.785}, e^{0.786}, e^{0.28}, e^{1.286}]\}^{\lambda} \implies e^{0.285} \leq e^{1.286\lambda}$$

which gives $\lambda = 0.22$ where $\lambda \in (0, \frac{1}{2})$.

CASE-III
Let $\psi, \phi \in (\frac{1}{2}, 1]$, then $S_m(\psi, \phi, \sigma) = e^{|\psi-\sigma|+|\phi-\sigma|}$
Take $\psi = \frac{1}{4}$ and $\phi = \frac{1}{2}$ then $M(\frac{1}{4}) = \frac{7}{20}, K(\frac{1}{4}) = \frac{1}{2}, H(\frac{1}{5}) = \frac{5}{14}$ and $J(\frac{1}{5}) = \frac{1}{2}$
substitute the above values in **(3.1.2)**

$$S_m(\frac{7}{20}, 0\frac{7}{20}, \frac{5}{14}) \leq$$
$$\Big\{max[S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}), S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}),$$
$$S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2}), S_m(\frac{7}{20}, \frac{7}{20}, \frac{1}{2})S_m(\frac{5}{14}, \frac{5}{14}, \frac{1}{2})]\Big\}^\lambda$$

which implies that

$$e^{0.014} \leq \Big\{ \max[e^{0.3}e^{0.28}, e^{0.3}e^{0.28}, e^{0.3}e^{0.28}, e^{0.3}e^{0.28}] \Big\}^\lambda$$

$$e^{0.014} \leq \{ \max[e^{0.58}, e^{0.58}, e^{0.58}, e^{0.58}]\}^\lambda \implies e^{0.014} \leq e^{0.5.8\lambda}$$

this gives that $\lambda = 0.14$ where $\lambda \in (0, \frac{1}{2})$.

Hence the inequality **(3.3.2)** holds.

It can be seen that "$\frac{1}{2}$" is a unique common fixed point for four self mappings M, K H, and J.

4 Conclusions

In this article, we established a common fixed point theorem in S_m-metric space by using weakly-compatible mappings, semi-compatible mappings, and reciprocally continuous mappings for four self-maps. Furthermore, our results are also justified with suitable examples.

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