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Closed Isometric Linear Transformations of Complex Spacetime endowed with Euclidean, or Lorentz, or generally Isotropic Metric

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Abstract

This paper is the first in a series of documents showing that *Newtonian physics* and *Einsteinian relativity theory* can be unified, by using a Generalized Real Boost (GRB), which expresses both the *Galilean Transformation* (GT) and the *Lorentz Boost*. Here, it is proved that the Closed Linear Transformations (CLTs) in Spacetime (ST) correlating frames having parallel spatial axes, are expressed via a 4x4 matrix Λ_I, which contains complex Cartesian Coordinates (CCs) of the velocity of one Observer / Frame (O/F) wrt another. In the case of generalized Special Relativity (SR), the inertial Os/Fs are related via isotropic ST endowed with constant real metric, which yields the constant characteristic parameter ω_I that is contained in the CLT and GRB of the specific SR. If ω_I is imaginary number, the ST can only be described by using complex CCs and there exists real Universal Speed (c_I). The specific value $\omega_I=\pm i$ gives the *Lorentzian-Einsteinian versions* of CLT and GRB in ST endowed with metric: $-g_{I00}\eta$ and $c_I=c$, where i; c; g_{I00}; η are the imaginary unit; speed of light in vacuum; time-coefficient of metric; *Lorentz metric*, respectively. If ω_I is real number, the corresponding ST can be described by using real CCs, but does not exist c_I. The specific value $\omega_I=0$ gives GT with infinite c_I. GT is also the reduction of the CLT and GRB, if one O/F has small velocity wrt another. The results may be applied to any ST

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endowed with isotropic metric, whose elements (four-vectors) have spatial part (vector) that is element of the ordinary *Euclidean space*.

Keywords: 5th Euclidean postulate; complex space; electromagnetic tensor; Euclidean metric; Euclidean space; Galilean Transformation, general relativity; isometry; linear transformation; Lorentz boost, Lorentz metric, Lorentz transformation, Minkowski spacetime, Newtonian physics, spacetime; special relativity; universal speed.

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Abbreviations-Annotations

CCs: Cartesian Coordinates
CILToCST: Closed Isometric Linear Transformation of Complex Spacetime
CLT: Closed Linear Transformation
c_{I}: Universal Speed
E^3: three-dimensional Euclidean Space
E^4: Euclidean Spacetime
ERT: Einsteinian Relativity Theory
ESR: Einsteinian Special Relativity
GR: General Relativity
GRB: Generalized Real Boost
GSR: Generalized Special Relativity
GT: Galilean Transformation
IO: Inertial Observer
LT: Linear Transformation
LB: Lorentz Boost
M^4: Minkowski Spacetime
NPs: Newtonian Physics
O/F: Observer / Frame
QMs: Quantum Mechanics
RT: Relativity Theory
SR: Special Relativity
ST: Spacetime (four-dimensional Space)
TPs: Theory of Physics
U: Invariant Speed
wrt: with respect to

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1 Introduction

Linear transformations (LTs) are very important in Relativity Theory (RT) and Quantum Mechanics (QMs) [1]. Moreover, there exist many different approaches of RT, which emerge the corresponding QMs. For instance, *Galilean Transformation* (GT) endowed with the corresponding metric of Spacetime (ST) produces *Newtonian Physics* (NPs), which gives the classic QMs (*Schrödinger Equation*). Thus, many low-velocity phenomena, like the atomic spectra (without fine structure) were explained. On the other hand, *Lorentz Transformation* (endowed with the *Lorentz metric* of ST) produces Einsteinian Special Relativity (ESR), which gives relativistic QMs (*Klein-Gordon Equation*). Thus, many high-velocity phenomena and the fine structure of atomic spectra were explained [2].

In this paper, we prove that there exist two types of Closed Isometric Linear Transformation of Complex Spacetime (CILToCST) with common solution the GT. These can apply not only to Special Relativity (SR), but also to General Relativity (GR), because they are reached without adopting one specific metric of spacetime. In addition, any complex *Cartesian Coordinates* (CCs) of the theory may be turned to the corresponding real CCs, in order to be perceived by human senses [3] (pp. 5-6).

SR relates the frames of Inertial Observers (IOs), via LTs of linear spacetime. ESR uses real spacetime (*Minkowski spacetime*) (M^4) endowed with *Lorentz Metric* (η) and the frames of two IOs with parallel spatial axes are always related via *Lorentz Boost* (LB). But is known that LB is not Closed Linear Transformation (CLT). In contrast, Lorentz Transformation (combination of *spatial Euclidean Rotation* with LB) is CLT (e.g. see [4], p. 41, eq. 1.104). Thus, if three Observers / Frames (Os/Fs): O*xyz*, O*′x′y′z′* and O*′′x′′y′′z′′* are related, where the axes of O*′x′y′z′* are parallel not only to the corresponding axes of O*xyz*, but also to the corresponding axes of O*′′x′′y′′z′′*, then the axes of O*xyz* and O*′′x′′y′′z′′* are not parallel (Figure 1). Thus, the transitive attribute in parallelism (which is equivalent to the *5^th^ Euclidean postulate*) is cancelled, when more than two Os/Fs are related. This consideration leads to successful results, such as *Thomas Precession*, which explains the fine structure of atomic spectra. But this happens only if we take successive observers O, O′ and O′′ with Thomas' order [5]. The reversed order of this sequence yields a result with 200% relative error.

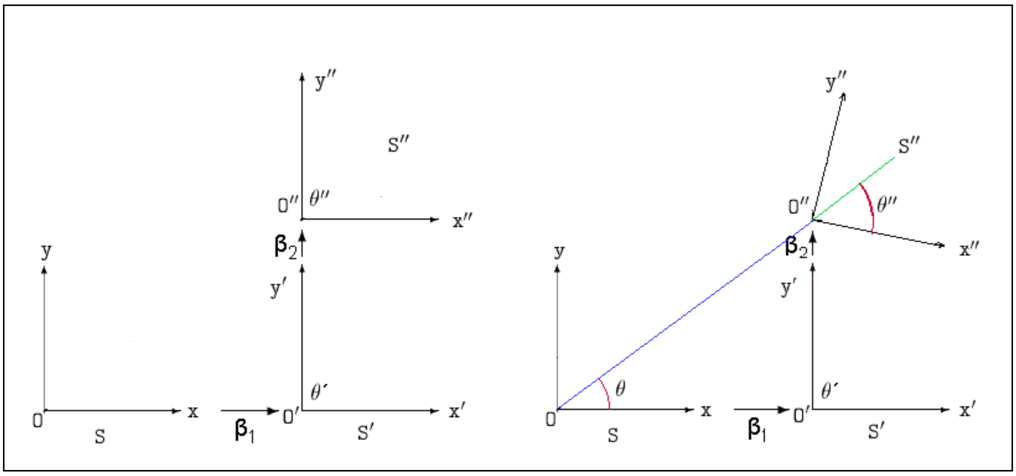

Figure 1: *Correlation of three successive observers (frames), by using Lorentz Boost. The frame O΄x΄y΄z΄ has parallel axes to the corresponding of frame Oxyz, moving with velocity ($\beta_1 c$, 0, 0) wrt Oxyz. The frame O΄΄x΄΄y΄΄z΄΄ has parallel axes to the corresponding of frame O΄x΄y΄z΄, moving with velocity (0, $\beta_2 c$, 0) wrt O΄x΄y΄z΄. The correlation of the observers, by using Lorentz Boost, cancels the absolute character of parallelism. Thus, the axes of frame O΄΄x΄΄y΄΄z΄΄ are not parallel to the corresponding of frame Oxyz (Thomas Rotation).*

In this paper, we prove that there exists CLT, which relates Os/Fs with parallel spatial axes (in case of IOs, or observers that have the same acceleration). Thus, the transitive attribute in parallelism is valid in complex three-dimensional Euclidean Space (E^3) and the axes rotation that happens in real space, when more than two observers are related, is the equivalent phenomenon of the corresponding Generalized Real Boost (GRB) [3] (pp. 5-6). The CLT is divided into *two cases*: *one*, where time depends on the position where the event happens, which can have real Invariant Speed (U) and *another*, where time is independent from the position and has $U=\infty$. Moreover, the demand that the CLT is isometric, gives the CILToCST. If *the metric of ST is independent from the position* of the event in ST, we have the *case of SR* and the CILToCST may be applied globally, *relating IOs*. Thus, infinite number of SR-theories is produced (each one of which with the corresponding metric of ST), keeping the ESR-formalism. In the case that *the metric of ST depends on the position* of the event in ST, we have the case of GR and the CILToCST may be applied locally, relating *Os/Fs with the same acceleration*. Thus, infinite number of GR-theories is produced (each one of which with the corresponding metric of ST of IOs), all of them keeping Einsteinian GR-formalism. Of course, zero acceleration leads to the corresponding SR. Finally, we present the *improper isometric* LT in ST endowed with *Euclidean*, or *Lorentz*, or generally any *isotropic metric*.

2 The Matrix of Closed Linear Transformation of Complex Spacetime

Initially, we determine the matrix Λ of active interpretation of the CLT of complex ST endowed with any metric.

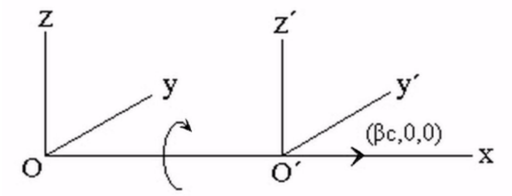

Figure 2: *Two frames Oxyz and O′x′y′z′ initially coincide. The second is moving with velocity (βc, 0, 0) wrt to Oxyz.*

2.1 Motion in the *x*-Direction

We consider one unmoved O/F O*xyz*, measuring real spacetime and another O/F O′*x*′*y*′*z*′ with parallel spatial axes, moving to the right, along *x*-axis with velocity $\upsilon = \left|\vec{\beta}\right| \mathrm{c} = \beta \mathrm{c}$ wrt O/F O*xyz* (Figure 2), where c=299,792,458 m s^{-1} is the speed of light in vacuum and the frames initially coincide. Supposing the next linear transformation

$$\mathrm{c}dt' = b\mathrm{c}dt + a dx + k dy + v dz \quad (1)$$

$$dx' = g\mathrm{c}dt + f dx + \delta dy + \theta dz \quad (2)$$

$$dy' = g_1\mathrm{c}dt + f_1 dx + h dy + \lambda dz \quad (3)$$

$$dz' = g_2\mathrm{c}dt + f_2 dx + \xi dy + \mu dz, \quad (4)$$

we determine the coefficients with the following condition: the space has isotropy. Rotating the coordinates system about the *x*-axis, by one negative right angle (Figure 1), we correspond the new axes to the initial axes: $t\to t$, $t'\to t'$, $x\to x$, $x'\to x'$, $y\to -z$, $y'\to -z'$, $z\to y$ and $z'\to y'$. Thus, from (1), we have

$$\mathrm{c}dt' = b\mathrm{c}dt + a dx - k dz + v dy. \quad (5)$$

(1) compared to (5), gives $k=v=0$. Besides, from (2) we have

$$dx' = g\mathrm{c}dt + f dx - \delta dz + \theta dy. \quad (6)$$

(2) compared to (6), gives $\delta=\theta=0$. Besides, from (3) we obtain

$$-dz' = g_1\mathrm{c}dt + f_1 dx - h dz + \lambda dy. \quad (7)$$

(4) compared to (7), gives $g_2=-g_1$, $f_2=-f_1$, $\xi=-\lambda$ and $\mu=h$. Besides, from (4), we have

$$dy' = g_2\mathrm{c}dt + f_2 dx - \xi dz + \mu dy. \quad (8)$$

(3) compared to (8), gives $g_2=g_1$, $f_2=f_1$, $\xi=-\lambda$ and $\mu=h$. So, $k=v=\delta=\theta=g_1=g_2=f_1=f_2=0$, $\xi=-\lambda$, $\mu=h$ and the transformation becomes

$$\mathrm{c}\mathrm{d}t' = b\mathrm{c}\mathrm{d}t + a\mathrm{d}x \tag{9}$$
$$\mathrm{d}x' = g\mathrm{c}\mathrm{d}t + f\mathrm{d}x \tag{10}$$
$$\mathrm{d}y' = h\mathrm{d}y + \lambda\mathrm{d}z \tag{11}$$
$$\mathrm{d}z' = -\lambda\mathrm{d}y + h\mathrm{d}z. \tag{12}$$

Using matrices we have the *active interpretation* of the LT [4] (p. 6):

$$\begin{bmatrix} \mathrm{c}\mathrm{d}t' \\ \mathrm{d}x' \\ \mathrm{d}y' \\ \mathrm{d}z' \end{bmatrix} = \begin{bmatrix} b & a & 0 & 0 \\ g & f & 0 & 0 \\ 0 & 0 & h & \lambda \\ 0 & 0 & -\lambda & h \end{bmatrix} \cdot \begin{bmatrix} \mathrm{c}\mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix}, \tag{13}$$

or equivalently,

$$\mathrm{d}X' = \Lambda_{1(x)}\,\mathrm{d}X, \tag{14}$$

where the base and the coordinates are

$$\lfloor \vec{e}_\mu \rfloor = [\vec{e}_0 \quad \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3] \;;\; \mathrm{d}X = \begin{bmatrix} \mathrm{d}x^0 \\ \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}x^3 \end{bmatrix} = \begin{bmatrix} \mathrm{c}\mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix} \tag{15}$$

respectively. Besides, the velocities are related in the following way:

$$\upsilon'_{\mathrm{x}} = \frac{g\mathrm{c} + f\upsilon_x}{b\mathrm{c} + a\upsilon_x}\mathrm{c} \;;\; \upsilon'_{\mathrm{y}} = \frac{h\upsilon_y + \lambda\upsilon_z}{b\mathrm{c} + a\upsilon_x}\mathrm{c} \;;\; \upsilon'_{\mathrm{z}} = \frac{-\lambda\upsilon_y + h\upsilon_z}{b\mathrm{c} + a\upsilon_x}\mathrm{c}. \tag{16}$$

2.2 General Linear Transformation (Motion in a random direction)

We then consider one unmoved O/F Oxyz and another O/F O$'x'y'z'$ with parallel spatial axes, moving with velocity (υ_x, υ_y, υ_z) wrt Oxyz, where they initially coincide (Figure 3). We rotate Oxyz, in order to parallelize the unitary vector $\hat{x}$ to the velocity vector $\vec{\upsilon}$ of the moving O$'x'y'z'$. This is sequentially achieved as following (Figure 4). We firstly rotate the coordinate system Oxyz about z-axis, through an angle θ: $\mathrm{O}(\hat{x},\hat{y},\hat{z}) \to \mathrm{O}(\hat{\imath},\hat{\jmath},\hat{k})$.We then rotate the coordinate system $\mathrm{O}(\hat{\imath},\hat{\jmath},\hat{k})$ about $\hat{\jmath}$, by an angle ω: $\mathrm{O}(\hat{\imath},\hat{\jmath},\hat{k}) \to \mathrm{O}(\hat{\imath}',\hat{\jmath}',\hat{k}')$. Thus, we have the transformation

$$\begin{bmatrix} x_{\mathrm{R}} \\ y_{\mathrm{R}} \\ z_{\mathrm{R}} \end{bmatrix} = \begin{bmatrix} \cos\omega\cos\theta & \cos\omega\sin\theta & \sin\omega \\ -\sin\theta & \cos\theta & 0 \\ -\sin\omega\,\cos\theta & -\sin\omega\,\sin\theta & \cos\omega \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{17}$$

where

$$\sin\theta = \frac{\upsilon_y}{\sqrt{\upsilon_x^{\,2} + \upsilon_y^{\,2}}} \;;\; \cos\theta = \frac{\upsilon_x}{\sqrt{\upsilon_x^{\,2} + \upsilon_y^{\,2}}} \;;\; \sin\omega = \frac{\upsilon_z}{|\vec{\upsilon}|} \;;\; \cos\omega = \frac{\sqrt{\upsilon_x^{\,2} + \upsilon_y^{\,2}}}{|\vec{\upsilon}|}. \tag{18}$$

As a result, the 3x3 matrix of (17) becomes

$$R=\begin{bmatrix} \dfrac{\beta_x}{|\vec{\beta}|} & \dfrac{\beta_y}{|\vec{\beta}|} & \dfrac{\beta_z}{|\vec{\beta}|} \\ -\dfrac{\beta_y}{\sqrt{\beta_x{}^2+\beta_y{}^2}} & \dfrac{\beta_x}{\sqrt{\beta_x{}^2+\beta_y{}^2}} & 0 \\ -\dfrac{\beta_x\beta_z}{|\vec{\beta}|\sqrt{\beta_x{}^2+\beta_y{}^2}} & -\dfrac{\beta_y\beta_z}{|\vec{\beta}|\sqrt{\beta_x{}^2+\beta_y{}^2}} & \dfrac{\sqrt{\beta_x{}^2+\beta_y{}^2}}{|\vec{\beta}|} \end{bmatrix} \tag{19}$$

and we define

$$\tilde{R}=\begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}. \tag{20}$$

The unit means that time is not affected by the spatial rotation. Moreover, the transformation $\mathrm{O}(\hat{x},\hat{y},\hat{z}) \to \mathrm{O}'(\hat{x},\hat{y},\hat{z})$ is analyzed to the following sequence of successive transformations:

$$\mathrm{O}(\hat{x},\hat{y},\hat{z}) \to \mathrm{O}(\hat{\imath}',\hat{\jmath}',\hat{k}')\,;\ \mathrm{O}(\hat{\imath}',\hat{\jmath}',\hat{k}') \to \mathrm{O}'(\hat{\imath}',\hat{\jmath}',\hat{k}')\,;$$
$$\mathrm{O}'(\hat{\imath}',\hat{\jmath}',\hat{k}') \to \mathrm{O}'(\hat{x},\hat{y},\hat{z})\,.$$

The above simple transformations have active interpretations:

$$X_{\mathrm{R}}=\tilde{R}X \quad ; \quad X'_{\mathrm{R}}=\Lambda_{1(x)}X_{\mathrm{R}} \quad ; \quad X'=\tilde{R}^{\mathrm{T}}X'_{\mathrm{R}}\,, \tag{21}$$

respectively, where $\tilde{R}^{\mathrm{T}}$ is the transpose matrix of $\tilde{R}$. Thus, the transformation $\mathrm{O}(\hat{x},\hat{y},\hat{z}) \to \mathrm{O}'(\hat{x},\hat{y},\hat{z})$ is actively interpreted:

$$\mathrm{d}X'=\tilde{R}^{T}\Lambda_{1(x)}\tilde{R}\,\mathrm{d}X=\Lambda_{(\beta)}\,\mathrm{d}X\,. \tag{22}$$

So, we calculate

$$\Lambda_{(\beta)}=\begin{bmatrix} b & \dfrac{a}{|\vec{\beta}|}\beta_x & \dfrac{a}{|\vec{\beta}|}\beta_y & \dfrac{a}{|\vec{\beta}|}\beta_z \\ \dfrac{g}{|\vec{\beta}|}\beta_x & (f-h)\dfrac{\beta_x{}^2}{|\vec{\beta}|^2}+h & (f-h)\dfrac{\beta_x\beta_y}{|\vec{\beta}|^2}+\dfrac{\beta_z\lambda}{|\vec{\beta}|} & (f-h)\dfrac{\beta_x\beta_z}{|\vec{\beta}|^2}-\dfrac{\beta_y\lambda}{|\vec{\beta}|} \\ \dfrac{g}{|\vec{\beta}|}\beta_y & (f-h)\dfrac{\beta_x\beta_y}{|\vec{\beta}|^2}-\dfrac{\beta_z\lambda}{|\vec{\beta}|} & (f-h)\dfrac{\beta_y{}^2}{|\vec{\beta}|^2}+h & (f-h)\dfrac{\beta_y\beta_z}{|\vec{\beta}|^2}+\dfrac{\beta_x\lambda}{|\vec{\beta}|} \\ \dfrac{g}{|\vec{\beta}|}\beta_z & (f-h)\dfrac{\beta_x\beta_z}{|\vec{\beta}|^2}+\dfrac{\beta_y\lambda}{|\vec{\beta}|} & (f-h)\dfrac{\beta_y\beta_z}{|\vec{\beta}|^2}-\dfrac{\beta_x\lambda}{|\vec{\beta}|} & (f-h)\dfrac{\beta_z{}^2}{|\vec{\beta}|^2}+h \end{bmatrix}. \tag{23}$$

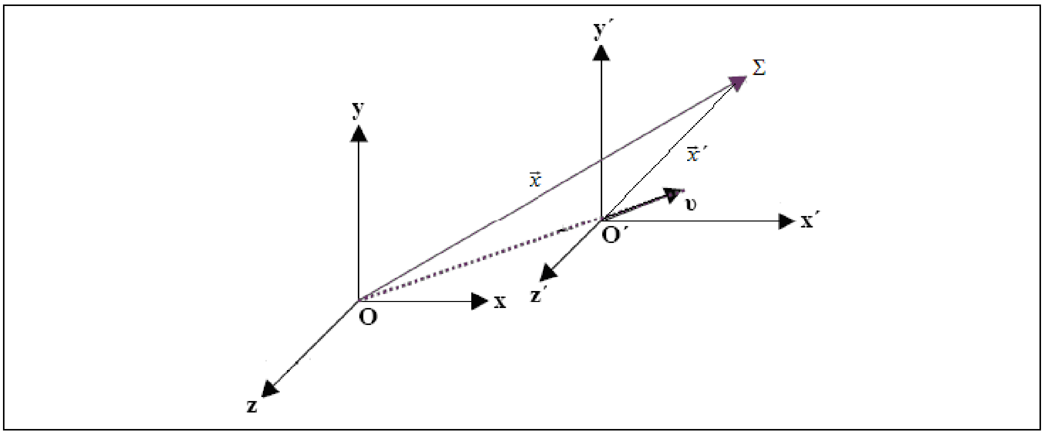

Figure 3: *Two frames Oxyz and O′x′y′z′, which initially coincide. The second is moving with random velocity (υ_x, υ_y, υ_z) wrt to Oxyz.*

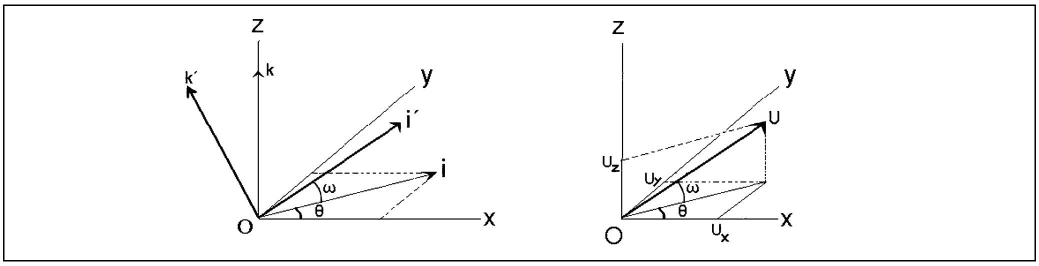

Figure 4: *Rotation of the initial frame Oxyz, in order to achieve parallelization of vector $\hat{x}$ to the velocity vector $\vec{\upsilon}$ of the moving observer O′x′y′z′ [$O(\hat{x},\hat{y},\hat{z}) \rightarrow O(\hat{i},\hat{j},\hat{k}) \rightarrow O(\hat{i}',\hat{j}',\hat{k}')$].*

2.3 ***Solution of the proper Closed Linear Transformation of Complex Spacetime (Correlation of two perpendicular moving Observers / Frames)***

We consider one unmoved O/F O*xyz*, another O/F O′*x′y′z′* with parallel spatial axes, moving to the right, along *x*-axis with velocity (βc, 0, 0) wrt O*xyz* and also a third O/F O′′*x′′y′′z′′* with parallel spatial axes, moving upward, along *y*-axis with velocity (0, βc, 0) wrt O*xyz* (Figure 5). All of them initially coincide and also $\beta > 0$, because

$$\beta = \left|\vec{\beta}\right|. \tag{24}$$

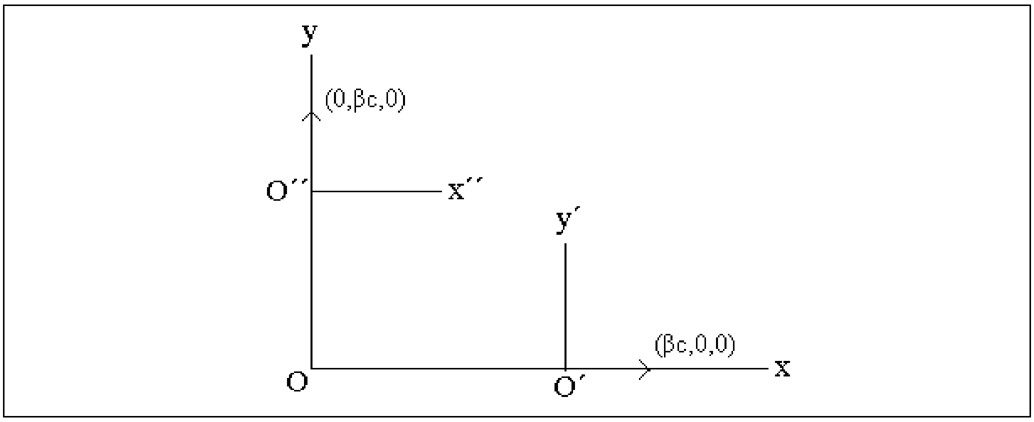

Figure 5: *Two frames O´x´y´z´ and O´´x´´y´´z´´ moving with corresponding velocities (βc, 0, 0) and (0, βc, 0) wrt Oxyz.*

The transformation $\mathrm{O}'(\hat{x},\hat{y},\hat{z}) \to \mathrm{O}''(\hat{x},\hat{y},\hat{z})$ is analyzed to the following sequence: $\mathrm{O}'(\hat{x},\hat{y},\hat{z}) \to \mathrm{O}(\hat{x},\hat{y},\hat{z})$; $\mathrm{O}(\hat{x},\hat{y},\hat{z}) \to \mathrm{O}''(\hat{x},\hat{y},\hat{z})$. The above simple transformations have active interpretations, respectively:

$$X = \Lambda_{1(x)}^{-1} X' \ \ ; \ \ X'' = \Lambda_{2(y)} X \ .$$

Thus, the transformation $\mathrm{O}'(\hat{x},\hat{y},\hat{z}) \to \mathrm{O}''(\hat{x},\hat{y},\hat{z})$ is actively interpreted:

$$X'' = \Lambda_{2(y)} \Lambda_{1(x)}^{-1} X' = \Pi X' .$$

According to equation (23), it is

$$\Lambda_{1(x)} = \begin{bmatrix} b & a & 0 & 0 \\ g & f & 0 & 0 \\ 0 & 0 & h & \lambda \\ 0 & 0 & -\lambda & h \end{bmatrix} \tag{25}$$

and

$$\Lambda_{2(y)} = \begin{bmatrix} b & 0 & a & 0 \\ 0 & h & 0 & -\lambda \\ g & 0 & f & 0 \\ 0 & \lambda & 0 & h \end{bmatrix}. \tag{26}$$

Thus, we have

$$\Pi = \Lambda_{2(y)}\Lambda_{1(x)}^{-1} = \Lambda_{2(y)} \cdot \begin{bmatrix} \frac{f}{bf-ag} & \frac{-a}{bf-ag} & 0 & 0 \\ \frac{-g}{bf-ag} & \frac{b}{bf-ag} & 0 & 0 \\ 0 & 0 & \frac{h}{h^2+\lambda^2} & \frac{-\lambda}{h^2+\lambda^2} \\ 0 & 0 & \frac{\lambda}{h^2+\lambda^2} & \frac{h}{h^2+\lambda^2} \end{bmatrix}. \tag{27}$$

or equivalently,

$$\Pi = \begin{bmatrix} \frac{bf}{bf-ag} & \frac{-ab}{bf-ag} & \frac{ah}{h^2+\lambda^2} & \frac{-a\lambda}{h^2+\lambda^2} \\ \frac{-gh}{bf-ag} & \frac{bh}{bf-ag} & \frac{-\lambda^2}{h^2+\lambda^2} & \frac{-h\lambda}{h^2+\lambda^2} \\ \frac{gf}{bf-ag} & \frac{-ag}{bf-ag} & \frac{fh}{h^2+\lambda^2} & \frac{-f\lambda}{h^2+\lambda^2} \\ \frac{-g\lambda}{bf-ag} & \frac{b\lambda}{bf-ag} & \frac{h\lambda}{h^2+\lambda^2} & \frac{h^2}{h^2+\lambda^2} \end{bmatrix}. \tag{28}$$

Now, we calculate the velocity factor $\vec{\beta}_4'$ of observer O$''x''y''z''$ wrt O$'x'y'z'$. Equation (16) can be applied, because O/F O$'x'y'z'$ is moving in the x-direction and observer O$''$ can be considered as the observed body. So, it is

$$\beta_{4x}' = \frac{g}{b}, \beta_{4y}' = \frac{h\beta}{b}, \ \beta_{4z}' = -\frac{\lambda\beta}{b} \tag{29}$$

and we obtain

$$\left|\vec{\beta}_4'\right| = \frac{\sqrt{g^2 + (h^2+\lambda^2)\beta^2}}{b} = \frac{\left|\vec{\beta}\right|}{b}\sqrt{\frac{g^2}{\left|\vec{\beta}\right|^2} + h^2 + \lambda^2} . \tag{30}$$

Replacing the above to (23), yields $\Lambda_{(\beta_4)} = \Lambda_4$. The condition that the transformation is closed, gives

$$\Pi = \Lambda_4. \tag{31}$$

Comparing the matrices, element by element, we shall calculate the parameters α, f and g. The transformation must be reduced to GT, if one IO has small velocity wrt another IO. So, it must be b, g, f, $h \neq 0$. We have two cases: (i) λ=0 and (ii) $\lambda \neq 0$.

2.3.1 *The case of proper Closed Linear Transformations of Complex Spacetime with λ=0 (time independent from the position, i.e. a=0).*

When λ=0, we compare matrices Π and Λ_4 element by element and we also take into account (29). Thus, we have: h_4=1 (from element Π_{33}) and λ_4=0 (from element Π_{13}). We then obtain f_4=1 (from element Π_{12}). So,

$$h_4=f_4=1 \;\; ; \;\; \lambda_4=0. \tag{32}$$

From elements Π_{10} and Π_{20}, we get

$$\frac{g_4 g}{|\vec{\beta}|\sqrt{\frac{g^2}{|\vec{\beta}|^2}+h^2}}=-\frac{gh}{bf-ag} \;\; ; \;\; \frac{g_4 h}{\sqrt{\frac{g^2}{|\vec{\beta}|^2}+h^2}}=\frac{gf}{bf-ag} \tag{33}$$

respectively. Thus,

$$g=-\frac{|\vec{\beta}|h^2}{f}. \tag{34}$$

From elements Π_{01} and Π_{02}, we have

$$\frac{a_4 g}{|\vec{\beta}|\sqrt{\frac{g^2}{|\vec{\beta}|^2}+h^2}}=-\frac{ab}{bf-ag} \;\; ; \;\; \frac{a_4 h}{\sqrt{\frac{g^2}{|\vec{\beta}|^2}+h^2}}=\frac{a}{h} \tag{35}$$

respectively. So,

$$\frac{g}{|\vec{\beta}|h}=-\frac{bh}{bf-ag}. \tag{36}$$

Replacing (34) to the above, implies

$$\alpha=0, \tag{37}$$

for the *CLT*, or g=0 for the *non-closed LT* (because (34) gives h=0 and matrix (25) cannot be identical). Thus, element Π_{11} gives

$$f=h \tag{38}$$

and (34) becomes

$$g=-|\vec{\beta}|h. \tag{39}$$

Finally, (23) yields the general *matrix of CLT:*

$$\Lambda_{(\beta)}=\begin{bmatrix} b & 0 & 0 & 0 \\ -h\beta_x & h & 0 & 0 \\ -h\beta_y & 0 & h & 0 \\ -h\beta_z & 0 & 0 & h \end{bmatrix}=\begin{bmatrix} b & \mathrm{O}^{\mathrm{T}} \\ -h\beta & h\mathrm{I}_3 \end{bmatrix} \;\; ; \;\; \beta=\begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix}=\begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix} \;\; ; \;\; \mathrm{O}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{40}$$

and the *typical matrix CLT (along x-axis):*

$$\Lambda_{(x)}=\begin{bmatrix} b & 0 & 0 & 0 \\ -h\beta & h & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & h \end{bmatrix}, \tag{41}$$

where $b=b_{(\beta)}$ and $h=h_{(\beta)}$.

Next, we calculate the corresponding CILToCST. The representation of the non-degenerate inner product in basis $\lfloor\vec{e}_\mu\rfloor=\lfloor\vec{e}_0 \quad \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3\rfloor=[\hat{ct},\hat{x},\hat{y},\hat{z}]$ is the real *matrix of metric*

$$g=\begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix}. \tag{42}$$

In this paper, we consider $g_{00}<0$ [signature of spacetime: (-+++), or (----)]. The fundamental equation of isometry - *Killing's equation* in a linear space - (see e.g. [4], p.10, eq.1.15) is

$$g'=\Lambda^{\mathrm{T}} g \Lambda. \tag{43}$$

The element by element comparison of the above matrices gives

$$g_{11}=g_{22}=g_{33}=g_{ii}=g'_{ii}=0,\ \ g'_{00}=b^2 g_{00}. \tag{44}$$

The isometry of spacetime [see e.g. [4], (p. 240)] is

$$\mathrm{d}S'^2=\mathrm{d}S^2, \tag{45}$$

or equivalently,

$$g'_{00}\,\mathrm{c}^2\mathrm{d}t'^2+\mathrm{d}x'_i g'_{ij}\,\mathrm{d}x'^j=g_{00}\,\mathrm{c}^2\mathrm{d}t^2+\mathrm{d}x_i g_{ij}\,\mathrm{d}x^j, \tag{46}$$

which combined with (44) and (40) gives

$$b\text{=1 for the } CLT, \tag{47}$$

or $b=-1$, $\pm$i for the *non-closed LT* (because matrix (25) cannot be identical). So, since b=1, *CLT* keeps *time invariant*. The *Einstein's summation convention* [4] (p. 3) was used in (46) and will be used in the equations that follow. Besides, (44ii) becomes

$$g'_{00}=g_{00}. \tag{48}$$

Thus, for any O/F the *metric of the ST* in accordance with the complex LT is

$$g_\Gamma=\begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{49}$$

We observe that $\det g_\Gamma=0$. So, this *spacetime is degenerate* [6] (p. 174). In order to calculate function h, we consider the unmoved O/F Oxyz, another O/F O$'x'y'z'$ moving to the right, along the x-axis with velocity $(\beta\mathrm{c},0,0)$ wrt Oxyz and a third O/F O$''x''y''z''$ moving to the left, along the x-axis with velocity $(-\beta\mathrm{c},0,0)$ wrt Oxyz. Thus, $X'=\Lambda_{(x)(\beta)}X$ and $X''=\Lambda_{(x)(-\beta)}X$ give

$$X''=\Lambda_{(x)(-\beta)}\Lambda^{-1}_{(x)(\beta)}X'. \tag{50}$$

Also, the *typical transformation of velocities* (16) becomes

$$\upsilon'_x = h\left(-\beta \mathrm{c} + \upsilon_x\right),\ \upsilon'_y = h\upsilon_y\,,\ \upsilon'_z = h\upsilon_z\,. \tag{51}$$

Thus, the calculation of the velocity factor of observer O′′ wrt O/F O$x'y'z'$ gives

$$\beta'_{3x} = -2h\beta\,,\ \beta'_{3y} = 0\,,\ \beta'_{3z} = 0\,. \tag{52}$$

As the transformation is closed, we have

$$\Lambda_{(x)(-\beta)}\Lambda^{-1}_{(x)(\beta)} = \Lambda_{(x)(\beta_3)}\,, \tag{53}$$

or equivalently,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta h & h & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & h \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta & \dfrac{1}{h} & 0 & 0 \\ 0 & 0 & \dfrac{1}{h} & 0 \\ 0 & 0 & 0 & \dfrac{1}{h} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2h_3 h\beta & h_3 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_3 \end{bmatrix}, \tag{54}$$

from which it derives that $h_3 = h_{(\beta_3)} = 1$ for any value of β. As h depends only on the norm of velocity factor β, the only solution is h=1. Hence, there derives the GT, which is expressed by the general matrix

$$\Lambda_{\Gamma(\beta)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta_x & 1 & 0 & 0 \\ -\beta_y & 0 & 1 & 0 \\ -\beta_z & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{O}^{\mathrm{T}} \\ -\beta & \mathrm{I}_3 \end{bmatrix}\ ;\ \beta = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix} = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix}, \tag{55}$$

and typical matrix along x-axis

$$\Lambda_{\Gamma(x)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{56}$$

which produces NPs with *invariant time* and *infinite universal speed*. As unmoved O/F Oxyz measures *real velocity*, the transformation matrix (Λ_Γ) contains only real numbers. So, the spacetime is limited to the *real domain* R^4. Moreover, this ST (*Galilean spacetime*) endowed with the *Galilean metric* (49), is degenerate.

2.3.2 *The case of proper Closed Linear Transformations of Complex Spacetime with λ≠0 (time dependent on the position).*

When $\lambda \neq 0$, we compare matrices Π and Λ_4 element by element and we also take into account (29). We start with $(\Lambda_4)_{21}+(\Lambda_4)_{12}=\Pi_{21}+\Pi_{12}$ and we obtain

$$\frac{2gh(f_4-h_4)}{\left|\vec{\beta}\right|\left(\frac{g^2}{\left|\vec{\beta}\right|^2}+h^2+\lambda^2\right)}=-\frac{ag}{bf-ag}-\frac{\lambda^2}{h^2+\lambda^2}\,. \tag{57}$$

We then use $(\Lambda_4)_{32}+(\Lambda_4)_{23}=\Pi_{32}+\Pi_{23}$, which gives

$$\frac{2h(f_4-h_4)}{\frac{g^2}{\left|\vec{\beta}\right|^2}+h^2+\lambda^2}=\frac{f-h}{h^2+\lambda^2}\,. \tag{58}$$

The combination of (57) with the above equation implies

$$\frac{g(f-h)}{\left|\vec{\beta}\right|\left(h^2+\lambda^2\right)}=-\frac{ag}{bf-ag}-\frac{\lambda^2}{h^2+\lambda^2}\,. \tag{59}$$

Also, $(\Lambda_4)_{31}+(\Lambda_4)_{13}=\Pi_{31}+\Pi_{13}$ gives

$$\frac{-2g(f_4-h_4)}{\left|\vec{\beta}\right|\left(\frac{g^2}{\left|\vec{\beta}\right|^2}+h^2+\lambda^2\right)}=\frac{b}{bf-ag}-\frac{h}{h^2+\lambda^2}\,. \tag{60}$$

The combination of the above equation with (58) also gives

$$\frac{-g(f-h)}{\left|\vec{\beta}\right|\left(h^2+\lambda^2\right)}=\frac{bh}{bf-ag}-\frac{h^2}{h^2+\lambda^2}\,. \tag{61}$$

We then add (59) and (61) and get

$$f=h\ \ ;\ \ f_4=h_4. \tag{62}$$

Moreover, from $(\Lambda_4)_{11}=\Pi_{11}$ and $(\Lambda_4)_{00}=\Pi_{00}$, we have

$$h_4=\frac{bh}{bf-ag}=\frac{bf}{bf-ag}=b_4\,, \tag{63}$$

which combined with (62) gives

$$f=h=b. \tag{64}$$

We then use $(\Lambda_4)_{22}=\Pi_2$, and we obtain

$$h_4=\frac{fh}{h^2+\lambda^2}\,. \tag{65}$$

The combination of the above equation with (63) gives

$$\frac{b}{bf-ag}=\frac{f}{h^2+\lambda^2}\,. \tag{66}$$

Furthermore, $(\Lambda_4)_{01}=\Pi_{01}$ and $(\Lambda_4)_{02}=\Pi_{02}$ give respectively:

$$\frac{g a_4}{\left|\vec{\beta}\right|\sqrt{\frac{g^2}{\left|\vec{\beta}\right|^2}+h^2+\lambda^2}}=-\frac{ab}{bf-ag}\,; \tag{67}$$

$$\frac{a_4}{\sqrt{\frac{g^2}{\left|\vec{\beta}\right|^2}+h^2+\lambda^2}}=\frac{a}{h^2+\lambda^2}\,. \tag{68}$$

The substitution of (68) to (67) gives

$$\frac{g}{\left|\vec{\beta}\right|\left(h^2+\lambda^2\right)}=-\frac{b}{bf-ag}\,. \tag{69}$$

Moreover, the combination of the above equation with (66) gives

$$g = -\left|\vec{\beta}\right| h\,. \tag{70}$$

Finally, (66) combined with (64) and (70) yields

$$\alpha=\frac{\lambda^2}{h\left|\vec{\beta}\right|}\,. \tag{71}$$

The replacement of (64), (70), (71) and

$$\lambda=\omega\left|\vec{\beta}\right| b\,, \tag{72}$$

makes the general matrix (23) equivalent to

$$\Lambda=\begin{bmatrix} b & \frac{\lambda^2}{b\left|\vec{\beta}\right|^2}\beta_x & \frac{\lambda^2}{b\left|\vec{\beta}\right|^2}\beta_y & \frac{\lambda^2}{b\left|\vec{\beta}\right|^2}\beta_z \\ -b\beta_x & b & \frac{\beta_z\lambda}{\left|\vec{\beta}\right|} & -\frac{\beta_y\lambda}{\left|\vec{\beta}\right|} \\ -b\beta_y & -\frac{\beta_z\lambda}{\left|\vec{\beta}\right|} & b & \frac{\beta_x\lambda}{\left|\vec{\beta}\right|} \\ -b\beta_z & \frac{\beta_y\lambda}{\left|\vec{\beta}\right|} & -\frac{\beta_x\lambda}{\left|\vec{\beta}\right|} & b \end{bmatrix}=b\begin{bmatrix} 1 & \omega^2\beta_x & \omega^2\beta_y & \omega^2\beta_z \\ -\beta_x & 1 & \omega\beta_z & -\omega\beta_y \\ -\beta_y & -\omega\beta_z & 1 & \omega\beta_x \\ -\beta_z & \omega\beta_y & -\omega\beta_x & 1 \end{bmatrix}. \tag{73}$$

We also define

$$\beta=\begin{bmatrix}\beta_x\\ \beta_y\\ \beta_z\end{bmatrix}=\begin{bmatrix}\beta^1\\ \beta^2\\ \beta^3\end{bmatrix}\,;\quad \delta=\begin{bmatrix}\delta_x\\ \delta_y\\ \delta_z\end{bmatrix}=\begin{bmatrix}\delta^1\\ \delta^2\\ \delta^3\end{bmatrix}\,;\quad \mathrm{A}_{(\beta)}=\begin{bmatrix}0 & \beta_z & -\beta_y\\ -\beta_z & 0 & \beta_x\\ \beta_y & -\beta_x & 0\end{bmatrix}=\begin{bmatrix}0 & \beta^3 & -\beta^2\\ -\beta^3 & 0 & \beta^1\\ \beta^2 & -\beta^1 & 0\end{bmatrix}. \tag{74}$$

It is noted that the antisymmetric matrix $\mathrm{A}_{(\beta)}$ is related to the *cross product (external product)* [7] (p. 1048), because

$$\mathrm{A}_{(\beta)}\,\delta = \left[-\vec{\beta}\times\vec{\delta}\right]=\left[\vec{\delta}\times\vec{\beta}\right]. \tag{75}$$

Thus, the four-vectors of two observers are related, by using the *general Matrix*:

$$\Lambda_{(\omega,\vec{\beta})}=b\begin{bmatrix}1 & \omega^2\beta_x & \omega^2\beta_y & \omega^2\beta_z\\ -\beta_x & 1 & \omega\beta_z & -\omega\beta_y\\ -\beta_y & -\omega\beta_z & 1 & \omega\beta_x\\ -\beta_z & \omega\beta_y & -\omega\beta_x & 1\end{bmatrix}=b\begin{bmatrix}1 & \omega^2\beta^T\\ -\beta & \mathrm{I}_3+\omega\mathrm{A}_{(\beta)}\end{bmatrix}. \tag{76}$$

Besides, the *typical Matrix* along *x*-axis is

$$\Lambda_{(x)(\omega,\beta)}=b\begin{bmatrix}1 & \omega^2\beta & 0 & 0\\ -\beta & 1 & 0 & 0\\ 0 & 0 & 1 & \omega\beta\\ 0 & 0 & -\omega\beta & 1\end{bmatrix}. \tag{77}$$

So, the *proper Closed Linear Transformation of Complex Spacetime* (22) is

$$\begin{bmatrix}\mathrm{cd}t'\\ \mathrm{d}x'\\ \mathrm{d}y'\\ \mathrm{d}z'\end{bmatrix}=b\begin{bmatrix}1 & \omega^2\beta_x & \omega^2\beta_y & \omega^2\beta_z\\ -\beta_x & 1 & \omega\beta_z & -\omega\beta_y\\ -\beta_y & -\omega\beta_z & 1 & \omega\beta_x\\ -\beta_z & \omega\beta_y & -\omega\beta_x & 1\end{bmatrix}\cdot\begin{bmatrix}\mathrm{cd}t\\ \mathrm{d}x\\ \mathrm{d}y\\ \mathrm{d}z\end{bmatrix}. \tag{78}$$

The pure mathematical approach is simply obtained by replacing ct→x^0. Thus,

$$\begin{bmatrix}\mathrm{d}x'^0\\ \mathrm{d}x'^1\\ \mathrm{d}x'^2\\ \mathrm{d}x'^3\end{bmatrix}=b\begin{bmatrix}1 & \omega^2\beta^1 & \omega^2\beta^2 & \omega^2\beta^3\\ -\beta^1 & 1 & \omega\beta^3 & -\omega\beta^2\\ -\beta^2 & -\omega\beta^3 & 1 & \omega\beta^1\\ -\beta^3 & \omega\beta^2 & -\omega\beta^1 & 1\end{bmatrix}\cdot\begin{bmatrix}\mathrm{d}x^0\\ \mathrm{d}x^1\\ \mathrm{d}x^2\\ \mathrm{d}x^3\end{bmatrix}. \tag{79}$$

Below, we calculate the corresponding CILToCST. For simplicity reasons, when we write i (the imaginary unit), we mean ±i:

$$\mathrm{i}\rightarrow\pm\mathrm{i}\quad;\quad -\mathrm{i}\rightarrow\mp\mathrm{i}\,. \tag{80}$$

The combination of the fundamental equation of isometry (43) (the *Killing's equation* in a linear space) with the above, gives

$$g_{11}=g_{22}=g_{33}=g_{ii};\ \ g_{00}=\frac{g_{ii}}{\omega^2};\ \ g'_{00}=b^2(1+\omega^2\beta^2)g_{00}=b^2(1+\omega^2\left|\vec{\beta}\right|^2)\frac{g_{ii}}{\omega^2}; \tag{81}$$

$$g'_{ii}=b^2(1+\omega^2\left|\vec{\beta}\right|^2)g_{ii}\,. \tag{82}$$

So, for any O/F, the metric of spacetime in accordance with the CILToCST is isotropic:

$$g=\begin{bmatrix}g_{00} & 0 & 0 & 0\\ 0 & g_{ii} & 0 & 0\\ 0 & 0 & g_{ii} & 0\\ 0 & 0 & 0 & g_{ii}\end{bmatrix} \tag{83}$$

and also

$$\omega^2 = \frac{g_{ii}}{g_{00}} . \tag{84}$$

Thus, ω^2 is a real number. So, ω is a real or an imaginary number, which only depends on the metric of spacetime. Besides, the metrics of the spacetime of two observers (frames) Oxyz and O$'x'y'z'$, are related using the formulas

$$g' = b^2 (1 + \omega^2 |\vec{\beta}|^2) g ; \tag{85}$$

$$g'_{ii} = b^2 (1 + \omega^2 |\vec{\beta}|^2) g_{ii} . \tag{86}$$

So,

$$b^2 = \frac{g'_{ii}}{g_{ii}} \frac{1}{1 + \omega^2 |\vec{\beta}|^2} . \tag{87}$$

Using the well-known *Lorentz γ-factor* function

$$\gamma_{(\delta)} = \frac{1}{\sqrt{1 - \delta^{\mathrm{T}} \delta}} = \frac{1}{\sqrt{1 - \vec{\delta} \cdot \vec{\delta}}} = \frac{1}{\sqrt{1 - |\vec{\delta}|^2}} = \gamma_{(\vec{\delta})} , \tag{88}$$

equation (87) may be written as

$$b^2 = \frac{g'_{ii}}{g_{ii}} \gamma^2_{(\mathrm{i}\omega\vec{\beta})} . \tag{89}$$

Besides, the isometry of spacetime (45) combined with (89) and (78) gives

$$g'_{ii} = g_{ii} . \tag{90}$$

Thus, (89) gives $b^2 = \gamma^2_{(\mathrm{i}\omega\vec{\beta})}$ and we obtain

$$b = \gamma_{(\mathrm{i}\omega\beta)} > 0 . \tag{91}$$

Moreover, (81iii) gives

$$g'_{00} = g_{00} . \tag{92}$$

This means that the metric of ST must be affected in the same way for any O/F, in the case of CILToCST. Equivalently, the observers that are related must be IOs or must have the *same acceleration*. Thus, the metric of spacetime in accordance with the CILToCST is

$$g = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{ii} & 0 & 0 \\ 0 & 0 & g_{ii} & 0 \\ 0 & 0 & 0 & g_{ii} \end{bmatrix} = g_{ii} \begin{bmatrix} \frac{1}{\omega^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -g_{00} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 \\ 0 & 0 & 0 & -\omega^2 \end{bmatrix} . \tag{93}$$

In case of SR (the frames are moved with constant velocity / the observes are IOs), equation (84) becomes

$$\omega_{\mathrm{I}}^{\;2} = \frac{g_{\mathrm{I}ii}}{g_{\mathrm{I}00}}\,. \tag{94}$$

So, the time and space metric's coefficients are independent from the position and they are combined to produce ω_{I}, which is *the characteristic parameter of SR.* The *continuity of the metric of spacetime at the point* $\lambda=\omega=0$ and the matrices (49) and (93) gives

$$\lim_{\omega_{\mathrm{I}}\to 0} g_{\mathrm{I}ii} = \lim_{\omega\to 0} g_{ii} = 0\,. \tag{95}$$

We also observe that *if the metric's coefficients have the same sign* [signature of spacetime: (----)], then the characteristic parameter ω_{I} *is a real number*, in contrast with the case that *the coefficients of metric have different signs* [signature of spacetime: (-+++)], where the characteristic parameter ω_{I} *is an imaginary number*.

The representation of the non-degenerate inner product in basis $\left[\vec{e}_\mu\right]=\left[\vec{e}_0 \quad \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3\right]=[\hat{ct},\hat{x},\hat{y},\hat{z}]$ for IOs is the matrix

$$g_{\mathrm{I}} = \begin{bmatrix} g_{\mathrm{I}00} & 0 & 0 & 0 \\ 0 & g_{\mathrm{I}ii} & 0 & 0 \\ 0 & 0 & g_{\mathrm{I}ii} & 0 \\ 0 & 0 & 0 & g_{\mathrm{I}ii} \end{bmatrix} = g_{\mathrm{I}ii}\begin{bmatrix} \frac{1}{\omega_{\mathrm{I}}^{\;2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -g_{\mathrm{I}00}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\omega_{\mathrm{I}}^{\;2} & 0 & 0 \\ 0 & 0 & -\omega_{\mathrm{I}}^{\;2} & 0 \\ 0 & 0 & 0 & -\omega_{\mathrm{I}}^{\;2} \end{bmatrix}. \tag{96}$$

Generally, equation (78) gives the *active transformation* of O/F Oxyz to O/F O$'x'y'z'$ (if they are accelerated with the same acceleration):

$$\begin{bmatrix} \mathrm{c}\,\mathrm{d}t' \\ \mathrm{d}x' \\ \mathrm{d}y' \\ \mathrm{d}z' \end{bmatrix} = \gamma_{(\mathrm{i}\omega\beta)}\begin{bmatrix} 1 & \omega^2\beta_x & \omega^2\beta_y & \omega^2\beta_z \\ -\beta_x & 1 & \omega\beta_z & -\omega\beta_y \\ -\beta_y & -\omega\beta_z & 1 & \omega\beta_x \\ -\beta_z & \omega\beta_y & -\omega\beta_x & 1 \end{bmatrix}\cdot\begin{bmatrix} \mathrm{c}\,\mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix}. \tag{97}$$

The replacement c$t\to x^0$ gives the pure mathematical approach. Thus,

$$\begin{bmatrix} \mathrm{d}x'^0 \\ \mathrm{d}x'^1 \\ \mathrm{d}x'^2 \\ \mathrm{d}x'^3 \end{bmatrix} = \gamma_{(\mathrm{i}\omega\beta)}\begin{bmatrix} 1 & \omega^2\beta^1 & \omega^2\beta^2 & \omega^2\beta^3 \\ -\beta^1 & 1 & \omega\beta^3 & -\omega\beta^2 \\ -\beta^2 & -\omega\beta^3 & 1 & \omega\beta^1 \\ -\beta^3 & \omega\beta^2 & -\omega\beta^1 & 1 \end{bmatrix}\cdot\begin{bmatrix} \mathrm{d}x^0 \\ \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}x^3 \end{bmatrix}. \tag{98}$$

Using vectors, the above transformation becomes

$$\mathrm{c}\,\mathrm{d}t' = \gamma_{(\mathrm{i}\omega\vec{\beta})}\left(\mathrm{c}\,\mathrm{d}t + \omega^2\vec{\beta}\cdot\mathrm{d}\vec{x}\right)\;;\; \mathrm{d}\vec{x}' = \gamma_{(\mathrm{i}\omega\vec{\beta})}\left[\left(\mathrm{d}\vec{x} - \vec{\beta}\,\mathrm{c}\,\mathrm{d}t\right) - \omega\vec{\beta}\times\mathrm{d}\vec{x}\right]. \tag{99}$$

Moreover, the *general* and *typical matrices* of CILToCST are, respectively:

$$\Lambda_{(\omega,\beta)} = \gamma_{(\mathrm{i}\omega\beta)} \begin{bmatrix} 1 & \omega^2\beta_x & \omega^2\beta_y & \omega^2\beta_z \\ -\beta_x & 1 & \omega\beta_z & -\omega\beta_y \\ -\beta_y & -\omega\beta_z & 1 & \omega\beta_x \\ -\beta_z & \omega\beta_y & -\omega\beta_x & 1 \end{bmatrix} = \gamma_{(\mathrm{i}\omega\beta)} \begin{bmatrix} 1 & \omega^2\beta^T \\ -\beta & \mathrm{I}_3 + \omega\,\mathrm{A}_{(\beta)} \end{bmatrix}; \quad (100)$$

$$\beta = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix}; \quad \mathrm{A}_{(\beta)} = \begin{bmatrix} 0 & \beta_z & -\beta_y \\ -\beta_z & 0 & \beta_x \\ \beta_y & -\beta_x & 0 \end{bmatrix}; \quad \Lambda_{(x)(\omega,\beta)} = \gamma_{(\mathrm{i}\omega\beta)} \begin{bmatrix} 1 & \omega^2\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega\beta \\ 0 & 0 & -\omega\beta & 1 \end{bmatrix}. \quad (101)$$

The above matrices Λ have the following properties:

$$\Lambda_{(\omega,\mathrm{O})} = \mathrm{I}_4 \;; \quad \mathrm{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad (102)$$

$$\Lambda^{-1}_{(\omega,\beta)} = \Lambda_{(\omega,-\beta)} \;; \quad (103)$$

$$\det \Lambda_{(\omega,\beta)} = 1. \quad (104)$$

In case of SR, the matrices form a new *group* (which corresponds to *Lorentz group*) with elements

$$d = (\Lambda_{(\omega_1,\beta)}, B) \quad ; \quad B = \begin{bmatrix} b^0 \\ b^1 \\ b^2 \\ b^3 \end{bmatrix} = \begin{bmatrix} \mathrm{c}\,t_0 \\ b_x \\ b_y \\ b_z \end{bmatrix}, \quad (105)$$

and operation

$$d_1 * d_2 = (\Lambda_{(\omega_1,\beta_2)}\, \Lambda_{(\omega_1,\beta_1)}, \Lambda_{(\omega_1,\beta_2)} B_1 + B_2), \quad (106)$$

where:

$\underset{1}{b}^{\mu}$ is the μ-coordinate which is measured in O$'x'y'z'$, when all the coordinates x^{ν}, for ν=0, 1, 2, 3, in Oxyz are equal to zero and

$\underset{2}{b}^{\mu}$ is the μ-coordinate which is measured in O$''x''y''z''$, when all the coordinates x'^{ν}, for ν=0, 1, 2, 3, in O$'xyz$ are equal to zero. The above operation expresses the successive transformations:

$$\mathrm{O}(\hat{x},\hat{y},\hat{z}) \to \mathrm{O}'(\hat{x},\hat{y},\hat{z}) \;;\; \mathrm{O}'(\hat{x},\hat{y},\hat{z}) \to \mathrm{O}''(\hat{x},\hat{y},\hat{z}). \quad (107)$$

These have active interpretations:

$$X' = \Lambda_{(\omega_1,\beta_1)} X + B_1 \quad ; \quad X'' = \Lambda_{(\omega_1,\beta_2)} X' + B_2 . \quad (108)$$

respectively. Thus, the transformation $\mathrm{O}'(\hat{x},\hat{y},\hat{z}) \to \mathrm{O}''(\hat{x},\hat{y},\hat{z})$ is actively interpreted:

$$X'' = \Lambda_{(\omega_1,\beta_2)} \Lambda_{(\omega_1,\beta_1)} X + \Lambda_{(\omega_1,\beta_2)} B_1 + B_2 . \quad (109)$$

As ω^2 is a real number, we observe that we always have *real time*. Besides, the *norm of the position four-vector* for *Os/Fs with the same acceleration / the same metric of ST*, is the corresponding *invariant quantity*

$$\mathrm{d}S^2 = \mathrm{d}X^T g\,\mathrm{d}X = g_{00}\,\mathrm{c}^2\mathrm{d}t^2 + g_{ii}\,\mathrm{d}\vec{x}^2 = g_{ii}\left(\frac{1}{\omega^2}\mathrm{c}^2\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) = -g_{00}\left(-\mathrm{c}^2\mathrm{d}t^2 - \omega^2\,\mathrm{d}\vec{x}^2\right). \quad (110)$$

In the case of SR, the above equation becomes

$$\mathrm{d}S^2 = \mathrm{d}X^T g\,\mathrm{d}X = g_{100}\,\mathrm{c}^2\mathrm{d}t^2 + g_{1ii}\,\mathrm{d}\vec{x}^2 = g_{1ii}\left[\frac{1}{\omega_{\mathrm{I}}^2}\mathrm{c}^2\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right] = -g_{00}\left(-\mathrm{c}^2\mathrm{d}t^2 - \omega_{\mathrm{I}}^2\,\mathrm{d}\vec{x}^2\right). \quad (111)$$

If *ω is a real number* [the coefficients of metric of time and space have the same sign: signature of spacetime: (----)], then

$$\omega = \pm\sqrt{\frac{g_{ii}}{g_{00}}} = s \quad (112)$$

with $s \in \mathrm{R}$. Thus, the transformation matrix (*Λ*) contains only real numbers and the ST is limited to the *real domain* R^4. Finally, the four-vectors of two Os/Fs have *the same metric*

$$g = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{ii} & 0 & 0 \\ 0 & 0 & g_{ii} & 0 \\ 0 & 0 & 0 & g_{ii} \end{bmatrix} = g_{ii}\begin{bmatrix} \frac{1}{s^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -g_{00}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -s^2 & 0 & 0 \\ 0 & 0 & -s^2 & 0 \\ 0 & 0 & 0 & -s^2 \end{bmatrix} \quad (113)$$

and their CCs are related via *the matrix*:

$$\Lambda_{(s,\beta)} = \gamma_{(\mathrm{i}\xi\beta)}\begin{bmatrix} 1 & s^2\beta_x & s^2\beta_y & s^2\beta_z \\ -\beta_x & 1 & s\beta_z & -s\beta_y \\ -\beta_y & -s\beta_z & 1 & s\beta_x \\ -\beta_z & s\beta_y & -s\beta_x & 1 \end{bmatrix} = \gamma_{(\mathrm{i}s\beta)}\begin{bmatrix} 1 & s^2\beta^T \\ -\beta & \mathrm{I}_3 + s\,\mathrm{A}_{(\beta)} \end{bmatrix}. \quad (114)$$

The typical matrix along *x*-axis is

$$\Lambda_{(x)(s,\beta)} = \gamma_{(\mathrm{i}\xi\beta)}\begin{bmatrix} 1 & s^2\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & s\beta \\ 0 & 0 & -s\beta & 1 \end{bmatrix}. \quad (115)$$

If *ω is an imaginary number* [the coefficients of metric of time and space have different sign: signature of spacetime: (-+++)], then

$$\omega = \mathrm{i}\sqrt{\frac{g_{ii}}{-g_{00}}} = \xi\,\mathrm{i} \;\; ; \;\; \xi = \sqrt{\frac{g_{ii}}{-g_{00}}} \quad (116)$$

with $\xi \in \mathrm{R}_+$. Thus, the transformation matrix (*Λ*) contains complex numbers and the spacetime is represented by the *complex domain* $\mathrm{R}\times\mathrm{C}^3$. Finally, the four-vectors of two Os/Fs have *the same metric*

$$g=\begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{ii} & 0 & 0 \\ 0 & 0 & g_{ii} & 0 \\ 0 & 0 & 0 & g_{ii} \end{bmatrix}=g_{ii}\begin{bmatrix} -\frac{1}{\xi^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=-g_{00}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 \\ 0 & 0 & \xi^2 & 0 \\ 0 & 0 & 0 & \xi^2 \end{bmatrix} \quad (117)$$

and their CCs are related via *the matrix*:

$$\Lambda_{(\xi\mathrm{i},\beta)}=\gamma_{(\xi\beta)}\begin{bmatrix} 1 & -\xi^2\beta_x & -\xi^2\beta_y & -\xi^2\beta_z \\ -\beta_x & 1 & \mathrm{i}\xi\beta_z & -\mathrm{i}\xi\beta_y \\ -\beta_y & -\mathrm{i}\xi\beta_z & 1 & \mathrm{i}\xi\beta_x \\ -\beta_z & \mathrm{i}\xi\beta_y & -\mathrm{i}\xi\beta_x & 1 \end{bmatrix}=\gamma_{(\xi\beta)}\begin{bmatrix} 1 & -\xi^2\beta^T \\ -\beta & \mathrm{I}_3+\mathrm{i}\xi\,\mathrm{A}_{(\beta)} \end{bmatrix}. \quad (118)$$

Besides, the typical *Matrix* along *x*-axis is

$$\Lambda_{(x)(\xi\mathrm{i},\beta)}=\gamma_{(\xi\beta)}\begin{bmatrix} 1 & -\xi^2\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{i}\xi\beta \\ 0 & 0 & -\mathrm{i}\xi\beta & 1 \end{bmatrix}. \quad (119)$$

The substitution of (64), (70), (71) and (72) to (16) gives the *velocities typical transformation of CILToCST*:

$$\upsilon'_x=\frac{\upsilon_x-\beta\mathrm{c}}{\mathrm{c}+\omega^2\beta\upsilon_x}\mathrm{c} \;\; ; \;\; \upsilon'_y=\frac{\upsilon_y+\omega\beta\upsilon_z}{\mathrm{c}+\omega^2\beta\upsilon_x}\mathrm{c} \;\; ; \;\; \upsilon'_z=\frac{\upsilon_z-\omega\beta\upsilon_y}{\mathrm{c}+\omega^2\beta\upsilon_x}\mathrm{c}. \quad (120)$$

For the purpose of finding a possible Invariant Speed (U) for *Os/Fs with the same ω (or equivalently the same acceleration)*, we assume that a particle is moving to the right with velocity $\vec{\upsilon}=U\vec{e}_1$; $U>0$. So, we have

$$\upsilon'_x=\frac{U-\beta\mathrm{c}}{\mathrm{c}+\omega^2\beta U}\mathrm{c} \;\; ; \;\; \upsilon'_y=0 \;\; ; \;\; \upsilon'_z=0. \quad (121)$$

According to the *Euclidean metric in the ordinary space E^3*, the norm of U is

$$U^2=\left(\frac{U-\beta\mathrm{c}}{\mathrm{c}+\omega^2\beta U}\mathrm{c}\right)^2+0+0=\frac{U^2-2\beta\mathrm{c}U+\beta^2\mathrm{c}^2}{\mathrm{c}^2+\omega^4\beta^2U^2+2\omega^2\beta\mathrm{c}U}\mathrm{c}^2, \quad (122)$$

which may be written as

$$\left(\omega^4U^4-\mathrm{c}^4\right)\beta^2+2U\left(\omega^2\mathrm{c}U^2+\mathrm{c}^3\right)\beta=0. \quad (123)$$

So, we obtain

$$U^2=-\frac{\mathrm{c}^2}{\omega^2}, \quad (124)$$

or equivalently,

$$\omega^2=-\frac{\mathrm{c}^2}{U^2}. \quad (125)$$

Since *norm U* > 0, *ω is an imaginary number* ($\omega = \xi \mathrm{i}, \xi \in \mathrm{R}_+$) independent from the velocity (i.e. depends on the acceleration or equivalently the gravitation). Thus, we have

$$U = \frac{1}{\xi}\mathrm{c}\,. \tag{126}$$

So, it is

$$\gamma_{(\mathrm{i}\omega\beta)} = \frac{1}{\sqrt{1+\omega^2\beta^{\mathrm{T}}\beta}} = \frac{1}{\sqrt{1+\omega^2\vec{\beta}\cdot\vec{\beta}}} = \frac{1}{\sqrt{1-\xi^2\left|\vec{\beta}\right|^2}} = \gamma_{(\xi\vec{\beta})} = \frac{1}{\sqrt{1-\left(\frac{\vec{u}}{U}\right)^2}} = \gamma_{\left(\frac{\vec{u}}{U}\right)} \tag{127}$$

and (110) can also be written as

$$\mathrm{d}S^2 = \mathrm{d}X^T g\,\mathrm{d}X = g_{ii}\left[-U^2\,\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right] = -g_{00}\left[-\mathrm{c}^2\mathrm{d}t^2 + \left(\frac{\mathrm{c}}{U}\right)^2\mathrm{d}\vec{x}^2\right]. \tag{128}$$

In the case of *spacetime endowed with* constant *metric* (or equivalently IOs), equation (126) becomes

$$c_{\mathrm{I}} = \frac{1}{\xi_{\mathrm{I}}}\mathrm{c}\,. \tag{129}$$

and we obtain the *Universal Speed* (c_{I}) of the *specific SR*. Besides, we have

$$\gamma_{(\mathrm{i}\omega_{\mathrm{I}}\vec{\beta})} = \frac{1}{\sqrt{1+\omega_{\mathrm{I}}{}^2\beta^{\mathrm{T}}\beta}} = \frac{1}{\sqrt{1+\omega_{\mathrm{I}}{}^2\left|\vec{\beta}\right|^2}} = \frac{1}{\sqrt{1-\xi_{\mathrm{I}}{}^2\left|\vec{\beta}\right|^2}} = \gamma_{(\xi_{\mathrm{I}}\vec{\beta})} = \frac{1}{\sqrt{1-\left(\frac{\vec{u}}{c_{\mathrm{I}}}\right)^2}} = \gamma_{\left(\frac{\vec{u}}{c_{\mathrm{I}}}\right)} \tag{130}$$

and

$$\mathrm{d}S^2 = \mathrm{d}X^T g\,\mathrm{d}X = g_{\mathrm{I}ii}\left[-c_{\mathrm{I}}{}^2\,\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right] = -g_{00}\left[-\mathrm{c}^2\mathrm{d}t^2 + \left(\frac{\mathrm{c}}{c_{\mathrm{I}}}\right)^2\mathrm{d}\vec{x}^2\right]. \tag{131}$$

Now, let us find the corresponding *Euclidean* CILToCST. We initially define

$$\mathrm{d}X_\omega = \begin{bmatrix}\mathrm{d}X_\omega^0\\ \mathrm{d}x^1\\ \mathrm{d}x^2\\ \mathrm{d}x^3\end{bmatrix} = \begin{bmatrix}\frac{1}{\omega}\mathrm{c}\mathrm{d}t\\ \mathrm{d}x\\ \mathrm{d}y\\ \mathrm{d}z\end{bmatrix} = \begin{bmatrix}\frac{1}{\omega}\mathrm{d}x^0\\ \mathrm{d}x^1\\ \mathrm{d}x^2\\ \mathrm{d}x^3\end{bmatrix}\ ;\ \mathrm{d}X_\omega{}^0 = \frac{1}{\omega}\mathrm{d}x^0\,, \tag{132}$$

where x^0 ; X_ω^0 are the zeroth-coordinates, by using the bases

$$\left\lfloor\vec{e}_\mu\right\rfloor = \left[\vec{e}_0 \quad \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3\right]\ ;\ \left\lfloor\vec{e}_\mu\right\rfloor = \left[\vec{E}_0 \quad \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3\right] \tag{133}$$

of ST endowed with metric (93) and *Euclidean metric*, respectively. Thus,

$$\vec{e}_0\cdot\vec{e}_0 = g_{00}\ ;\ \vec{E}_0\cdot\vec{E}_0 = 1\,, \tag{134}$$

where dot "·" is *Euclidean inner product* [4] (p. 7). So, we understand that

$$\vec{E}_0 = \frac{\mathrm{i}}{\sqrt{-g_{00}}}\vec{e}_0 . \tag{135}$$

Then, the CILToCST (97) can be written as

$$\begin{bmatrix} \frac{1}{\omega}\mathrm{c}\,\mathrm{d}\,t' \\ \mathrm{d}\,x' \\ \mathrm{d}\,y' \\ \mathrm{d}\,z' \end{bmatrix} = \gamma_{(\mathrm{i}\omega\beta)} \begin{bmatrix} 1 & \omega\beta_x & \omega\beta_y & \omega\beta_z \\ -\omega\beta_x & 1 & \omega\beta_z & -\omega\beta_y \\ -\omega\beta_y & -\omega\beta_z & 1 & \omega\beta_x \\ -\omega\beta_z & \omega\beta_y & -\omega\beta_x & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\omega}\mathrm{c}\,\mathrm{d}\,t \\ \mathrm{d}\,x \\ \mathrm{d}\,y \\ \mathrm{d}\,z \end{bmatrix}, \tag{136}$$

or equivalently,

$$\mathrm{d}X'_{\omega} = \tilde{R}_{(\omega\beta)}\,\mathrm{d}X_{\omega}, \tag{137}$$

where

$$\tilde{R}_{(\omega\beta)} = \gamma_{(\mathrm{i}\omega\beta)} \begin{bmatrix} 1 & \omega\beta_x & \omega\beta_y & \omega\beta_z \\ -\omega\beta_x & 1 & \omega\beta_z & -\omega\beta_y \\ -\omega\beta_y & -\omega\beta_z & 1 & \omega\beta_x \\ -\omega\beta_z & \omega\beta_y & -\omega\beta_x & 1 \end{bmatrix} \;;\; \beta = \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_z \end{bmatrix} = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix}, \tag{138}$$

or equivalently,

$$\tilde{R}_{(\omega\beta)} = \gamma_{(\mathrm{i}\omega\beta)} \begin{bmatrix} 1 & \omega\beta^T \\ -\omega\beta & \mathrm{I}_3 + \omega\mathrm{A}_{(\beta)} \end{bmatrix} = \gamma_{(\mathrm{i}\omega\beta)}\{\mathrm{I}_4 + \omega \begin{bmatrix} 0 & \beta^T \\ -\beta & \mathrm{A}_{(\beta)} \end{bmatrix}\}. \tag{139}$$

The above matrix $\tilde{R}$ is a *rotation matrix* with the following properties:

$$\tilde{R}_{(\omega\mathrm{O})} = \mathrm{I}_4 \;;\; \mathrm{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{140}$$

$$\tilde{R}^{-1}_{(\omega\beta)} = \tilde{R}^{T}_{(\omega\beta)} = \tilde{R}_{(-\omega\beta)}, \tag{141}$$

$$\det \tilde{R}_{(\omega\beta)} = 1. \tag{142}$$

The corresponding typical matrix along the *x*-axis in *Euclidean spacetime* (E^4) is

$$\tilde{R}_{(x)(\omega\beta)} = \gamma_{(\mathrm{i}\omega\beta)} \begin{bmatrix} 1 & \omega\beta & 0 & 0 \\ -\omega\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega\beta \\ 0 & 0 & -\omega\beta & 1 \end{bmatrix}. \tag{143}$$

Now, using vectors the CILToCST of equation (136) becomes

$$\frac{\mathrm{c}\mathrm{d}\,t'}{\omega} = \gamma_{(\mathrm{i}\omega\vec{\beta})}(\frac{\mathrm{c}\mathrm{d}\,t}{\omega} + \omega\vec{\beta}\cdot\mathrm{d}\,\vec{x}) \;;\; \mathrm{d}\,\vec{x}' = \gamma_{(\mathrm{i}\omega\vec{\beta})}\left[\left(\mathrm{d}\,\vec{x} - \omega\vec{\beta}\frac{\mathrm{c}\mathrm{d}\,t}{\omega}\right) - \omega\vec{\beta}\times\mathrm{d}\,\vec{x}\right], \tag{144}$$

or equivalently,

$$\mathrm{d}X'^{0}_{\omega} = \gamma_{(\mathrm{i}\omega\vec{\beta})}(\mathrm{d}X_{\omega}{}^{0} + \omega\vec{\beta}\cdot\mathrm{d}\vec{x}) \;\; ; \;\; \mathrm{d}\vec{x}' = \gamma_{(\mathrm{i}\omega\vec{\beta})}\left[\left(\mathrm{d}\vec{x} - \omega\vec{\beta}\,\mathrm{d}X_{\omega}{}^{0}\right) - \omega\vec{\beta}\times\mathrm{d}\vec{x}\right]. \quad (145)$$

Thus, we have the *Euclidean metric* of the position four-vector X_ω in E^4, and the corresponding invariant quantity is

$$\mathrm{d}S_\omega^2 = \mathrm{d}X_\omega^T g_{\mathrm{E}}\,\mathrm{d}X_\omega = \frac{1}{\omega^2}\mathrm{c}^2\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 = \left(\mathrm{d}X_\omega{}^0\right)^2 + \mathrm{d}\vec{x}^2 = \frac{1}{g_{ii}}\mathrm{d}S^2, \qquad (146)$$

according to (110ii).
Also, we observe that *β-factor* can be written as

$$\beta^i = \frac{\mathrm{d}x^i}{\mathrm{d}x^0} = \frac{\mathrm{d}x^i}{\omega\,\mathrm{d}X_\omega{}^0} = \frac{B^i}{\omega} \;\; ; \;\; B^i = \frac{\mathrm{d}x^i}{\mathrm{d}X_\omega{}^0}, \qquad (147)$$

by using (132ii). The quantity B^i is called *B-factor* and it can substitute the *β-factor*, in E^4. Then, equations (136-139) are rewritten:

$$\begin{bmatrix} \mathrm{d}X'^{0}_{\omega} \\ \mathrm{d}x'^1 \\ \mathrm{d}x'^2 \\ \mathrm{d}x'^3 \end{bmatrix} = \gamma_{(\mathrm{i}B)}\begin{bmatrix} 1 & B^1 & B^2 & B^3 \\ -B^1 & 1 & B^3 & -B^2 \\ -B^2 & -B^3 & 1 & B^1 \\ -B^3 & B^2 & -B^1 & 1 \end{bmatrix}\cdot\begin{bmatrix} \mathrm{d}X_\omega{}^0 \\ \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}x^3 \end{bmatrix}, \qquad (148)$$

$$\mathrm{d}X'_\omega = \tilde{R}_{(B)}\,\mathrm{d}X_\omega, \qquad (149)$$

$$\tilde{R}_{(B)} = \gamma_{(\mathrm{i}B)}\begin{bmatrix} 1 & B^1 & B^2 & B^3 \\ -B^1 & 1 & B^3 & -B^2 \\ -B^2 & -B^3 & 1 & B^1 \\ -B^3 & B^2 & -B^1 & 1 \end{bmatrix} \;\; ; \;\; B = \begin{bmatrix} B^1 \\ B^2 \\ B^3 \end{bmatrix}, \qquad (150)$$

$$\tilde{R}_{(B)} = \gamma_{(\mathrm{i}B)}\begin{bmatrix} 1 & B^T \\ -B & \mathrm{I}_3 + \mathrm{A}_{(B)} \end{bmatrix} = \gamma_{(\mathrm{i}B)}\{I_4 + \begin{bmatrix} 0 & B^T \\ -B & \mathrm{A}_{(\beta)} \end{bmatrix}\}. \qquad (151)$$

The above matrix $\tilde{R}$ is a *rotation matrix* having the following properties:

$$\tilde{R}_{(\mathrm{O})} = \mathrm{I}_4 \;\; ; \qquad (152)$$

$$\tilde{R}^{-1}_{(B)} = \tilde{R}^T_{(B)} = \tilde{R}_{(-B)}\,; \qquad (153)$$

$$\det\tilde{R}_{(B)} = 1. \qquad (154)$$

Besides, the corresponding typical matrix along the *x*-axis in E^4 is

$$\tilde{R}_{(x)(B)} = \gamma_{(\mathrm{i}B)}\begin{bmatrix} 1 & B & 0 & 0 \\ -B & 1 & 0 & 0 \\ 0 & 0 & 1 & B \\ 0 & 0 & -B & 1 \end{bmatrix}. \qquad (155)$$

Note that the above transformation can be limited in the real spacetime (R^4), because the corresponding *Lorentz γ-factor* is positive for any real *B-factor*.

We observe that the above results could be obtained from the initial equations of E^4: (136-146), when $\omega\to 1$ and ct$\to X_\omega$. We also observe that $\tilde{R}$ reminds us of the *contravariant electromagnetic tensor* [3] (p. 14), [4] (p. 414):

$$F_{(E,B_{\mathrm{m}})}=\begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & \mathrm{c}\,B_{\mathrm{m}}{}^3 & -\mathrm{c}\,B_{\mathrm{m}}{}^2 \\ -E^2 & -\mathrm{c}\,B_{\mathrm{m}}{}^3 & 0 & \mathrm{c}\,B_{\mathrm{m}}{}^1 \\ -E^3 & \mathrm{c}\,B_{\mathrm{m}}{}^2 & -\mathrm{c}\,B_{\mathrm{m}}{}^1 & 0 \end{bmatrix}, \tag{156}$$

where E and B_{m} are the *intensity of electric field* and *induction of magnetic field*, respectively [4] (p. 396). Actually, they are correlated via the formula

$$\tilde{R}_{(B)}=F_{(E,B_{\mathrm{m}})}+\gamma_{(\mathrm{i}B)}\,\mathrm{I}_4 \;\;;\;\; E^j=\gamma_{(\mathrm{i}B)}B^j\,;\; B_{\mathrm{m}}{}^j=\frac{\gamma_{(\mathrm{i}B)}B^j}{\mathrm{c}}\;. \tag{157}$$

Thus, it is

$$E^j=\mathrm{c}\,B_{\mathrm{m}}{}^j \;\;;\;\; E^jE_j=\mathrm{c}^2\left(B_{\mathrm{m}}{}^jB_{\mathrm{m}\,j}\right) \tag{158}$$

where (158ii) is the same as the *electromagnetic waves in vacuum*, while (158i) means that the vectors of the induction of magnetic field and intensity of electric field are parallel. This reveals a hidden correlation between the *spacetime* and *electromagnetism* (*Maxwell equations*).

Moreover, for any *constant value of* ω_{I} (or more precisely for any *constant metric*, i.e. *constant values of* $g_{\mathrm{I}00}$ and $g_{\mathrm{I}ii}$), we have a specific CILToCST which correlates IOs and the corresponding *SR-theory*.

Furthermore, the limit $s\to s_{\mathrm{I}}\to 0$ in the equations (113-115) and their combination with (95) gives GT of complex spacetime with *infinite universal speed*. In the same way, the limit $\xi\to\xi_{\mathrm{I}}\to 0$ in the equations (117-119) and their combination with (95), gives again GT. Thus, the result when λ=0 (GT) is embedded to the case when $\lambda\neq 0$, if we take the corresponding limit to zero ($\lambda\to 0$, or equivalently, $\omega\to 0$).

Besides, if one O/F has small velocity wrt another, the CILToCST (even been complex) is reduced to GT.

The replacement $\xi\to\xi_{\mathrm{I}}=1$ to the equations (118) and (119), produces the *Lorentzian-Einsteinian* version of CILToCST (Λ_{B}) [7] (pp.1047-1048), which is expressed via the general matrix

$$\Lambda_{\mathrm{B}(\beta)}=\gamma\begin{bmatrix} 1 & -\beta_x & -\beta_y & -\beta_z \\ -\beta_x & 1 & \mathrm{i}\beta_z & -\mathrm{i}\beta_y \\ -\beta_y & -\mathrm{i}\beta_z & 1 & \mathrm{i}\beta_x \\ -\beta_z & \mathrm{i}\beta_y & -\mathrm{i}\beta_x & 1 \end{bmatrix} \tag{159}$$

and the typical matrix along the x-axis

$$\Lambda_{\mathrm{B}(x)(\beta)} = \gamma \begin{bmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{i}\beta \\ 0 & 0 & -\mathrm{i}\beta & 1 \end{bmatrix}. \tag{160}$$

From (96), we take the corresponding metric of complex spacetime

$$g_{\mathrm{B}} = g_{\mathrm{I}ii} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = g_{\mathrm{I}ii}\eta\,, \tag{161}$$

which for $g_{\mathrm{I}ii}=1$ becomes the *Lorentz metric*. Thus, we have the SR-theory with *universal speed* being the *speed of light in vacuum* (c_{I}=c) [7,8]. This theory gives results that are exactly the same as ESR, when only two Os/Fs are related. But the results are different, when more than two Os/Fs are related. Besides, it calculates the fine structure peeks of atomic hydrogen's spectrum [8] (p. 4) more accurately than ESR. The explicit form of forward *Lorentzian-Einsteinian* CILToCST is

$$\mathrm{c}t' = \gamma(\mathrm{c}t - \beta_x x - \beta_y y - \beta_z z) \tag{162}$$
$$x' = \gamma(-\beta_x\, \mathrm{c}t + x + \mathrm{i}\beta_z y - \mathrm{i}\beta_y z) \tag{163}$$
$$y' = \gamma(-\beta_y \mathrm{c}t - \mathrm{i}\beta_z\, x + y + \mathrm{i}\beta_x z) \tag{164}$$
$$z' = \gamma(-\beta_z \mathrm{c}t + \mathrm{i}\beta_y x - \mathrm{i}\beta_x y + z) \tag{165}$$

The explicit form of reverse *Lorentzian-Einsteinian* CILToCST is

$$\mathrm{c}t = \gamma(\mathrm{c}t' + \beta_x x' + \beta_y y' + \beta_z z') \tag{166}$$
$$x = \gamma(\beta_x\, \mathrm{c}t' + x' - \mathrm{i}\beta_z y' + \mathrm{i}\beta_y z') \tag{167}$$
$$y = \gamma(\beta_y \mathrm{c}t' + \mathrm{i}\beta_z\, x' + y' - \mathrm{i}\beta_x z') \tag{168}$$
$$z = \gamma(\beta_z \mathrm{c}t' - \mathrm{i}\beta_y x' + \mathrm{i}\beta_x y' + z') \tag{169}$$

When *the metric of ST depends on the position* of the event in spacetime (GR), the transformation is applied locally, not globally (correlating Os/Fs with the same acceleration / gravitation). A metric is in accordance with the CILToCST, if only the limit of vanishing acceleration leads to the corresponding SR. Thus, the usage of (84) and (94) leads to

$$\lim_{\vec{a}\to 0} g_{ii} = g_{\mathrm{I}ii}\,; \tag{170}$$

$$\lim_{\vec{a}\to 0} g_{00} = \lim_{\vec{a}\to 0} \frac{g_{ii}}{\omega^2} = \frac{g_{\mathrm{I}ii}}{\omega_{\mathrm{I}}{}^2} = g_{\mathrm{I}00}\,. \tag{171}$$

3 Proper Time – Special and General Relativity

Let P be a particle moving with velocity $\vec{\upsilon}_P$ wrt observer O ($\vec{\upsilon}'_P$ wrt observer O′) in spacetime. The generalized definition of proper time (τ) is

$$\mathrm{d}\tau^2 = \frac{\mathrm{d}S^2}{g_{00}\,\mathrm{c}^2}. \tag{172}$$

Using (84) and (110), we have

$$\mathrm{d}\tau^2 = \frac{g_{ii}}{g_{00}\,\mathrm{c}^2}\left(\frac{1}{\omega^2}\mathrm{c}^2\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) = \frac{\omega^2}{\mathrm{c}^2}\left(\frac{1}{\omega^2}\mathrm{c}^2\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right), \tag{173}$$

or equivalently,

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 + \frac{\omega^2}{\mathrm{c}^2}\mathrm{d}\vec{x}^2 = \mathrm{d}t^2\left(1 + \frac{\omega^2}{\mathrm{c}^2}\vec{\upsilon}_P{}^2\right). \tag{174}$$

Thus, the relation between the time and the proper time is

$$\frac{\mathrm{d}t^2}{\mathrm{d}\tau^2} = \gamma^2_{(\mathrm{i}\omega\beta_P)}. \tag{175}$$

For $\omega = s$ with $s \in \mathrm{R}$, there does not exists real Invariant Speed (U) and the γ-factor is always positive. So,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_{(\mathrm{i}s\beta_P)}. \tag{176}$$

When $\omega = \xi\mathrm{i}$ with $\xi \in R$, there exists a real U. If the speed of particle is less than the invariant speed ($|\vec{\upsilon}_P| < U$), then the γ-factor is positive again. Thus,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_{(\xi\beta_P)}. \tag{177}$$

For GT with $s \to \xi \to 0$ (the limit of degenerate spacetime), $\gamma_{(\mathrm{i}s\beta_P)} = \gamma_{(\xi\beta_P)} = 1$. So, d$\tau$ = dt' = dt (time is invariant) as we know in NPs.

In the case of ST with constant metric (IOs), equation (175) becomes

$$\frac{\mathrm{d}t^2}{\mathrm{d}\tau^2} = \gamma^2_{(\mathrm{i}\omega_\mathrm{I}\beta_P)}. \tag{178}$$

Thus, the *Lorentzian-Einsteinian* version of CILToCST with $\omega_\mathrm{I} = \mathrm{i}$ ($\xi_\mathrm{I} = 1$), gives $U = c_\mathrm{I} = \mathrm{c}$. If the speed of a particle is less than the *speed of light in vacuum* ($|\vec{\upsilon}_P| < \mathrm{c}$), then γ-factor is positive again. Thus,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_{(\beta_P)}. \tag{179}$$

and we have the same result as the ESR.

In any case, using proper time, we can define four-velocity, four-momentum etc, building the whole structure of Generalized SR and GR.

4 The Results of Closed Linear Transformation of Complex Spacetime - Discussion

In this section, we present the typical matrix $\Lambda_{(x)(\beta)}$, the general matrix $\Lambda_{(\beta)}$, the covariant matrix of spacetime metric g, the invariant speed U and the domain of the coordinates C^4 that corresponds to the transformation of a contravariant infinitesimal four-vector in spacetime: $dX'=\Lambda dX$.

$(\Lambda_{(x)(\beta)}, \Lambda_{(\beta)}, g, U, C^4)$

$\lambda = \omega|\vec{\beta}|b = ?$

$\lambda \neq 0$

$$\Lambda_{(x)(\beta)} = b_{(\beta)}\begin{bmatrix} 1 & \omega^2\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega\beta \\ 0 & 0 & -\omega\beta & 1 \end{bmatrix},$$

$$\Lambda_{(\beta)} = b_{(\beta)}\begin{bmatrix} 1 & \omega^2\beta^T \\ -\beta & I_3+\omega A_{(\beta)} \end{bmatrix},\ U \in \{R, I\}.$$

isometry

$$\Lambda_{(x)(\omega,\beta)} = \gamma_{(i\omega\beta)}\begin{bmatrix} 1 & \omega^2\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega\beta \\ 0 & 0 & -\omega\beta & 1 \end{bmatrix},$$

$$\Lambda_{(\beta)} = \gamma_{(i\omega\beta)}\begin{bmatrix} 1 & \omega^2\beta^T \\ -\beta & I_3+\omega A_{(\beta)} \end{bmatrix},$$

$$g = -g_{00}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 \\ 0 & 0 & 0 & -\omega^2 \end{bmatrix},$$

$\omega = ?$

$\lambda = 0$

$$\Lambda_{(x)(\beta)} = \begin{bmatrix} b_{(\beta)} & 0 & 0 & 0 \\ -h_{(\beta)}\beta & h_{(\beta)} & 0 & 0 \\ 0 & 0 & h_{(\beta)} & 0 \\ 0 & 0 & 0 & h_{(\beta)} \end{bmatrix},$$

$$\Lambda_{(\beta)} = \begin{bmatrix} b_{(\beta)} & O^T \\ -h_{(\beta)}\beta & h_{(\beta)} I_3 \end{bmatrix},\ U = +\infty.$$

isometry

$$\Lambda_{\Gamma(x)(\beta)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Lambda_{\Gamma(\beta)} = \begin{bmatrix} 1 & O^T \\ -\beta & I_3 \end{bmatrix},$$

$$g_\Gamma = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ U = +\infty,\ R^4.$$

$\omega=\xi\mathrm{i}\in\mathrm{I}$	$\omega=s\in\mathrm{R}$
$\Lambda_{(x)(\mathrm{i}\xi,\beta)}=\gamma_{(\xi\beta)}\begin{bmatrix}1 & -\xi^2\beta & 0 & 0\\ -\beta & 1 & 0 & 0\\ 0 & 0 & 1 & \mathrm{i}\xi\beta\\ 0 & 0 & -\mathrm{i}\xi\beta & 1\end{bmatrix}$,	$\Lambda_{(x)(\xi,\beta)}=\gamma_{(\mathrm{i}s\beta)}\begin{bmatrix}1 & s^2\beta & 0 & 0\\ -\beta & 1 & 0 & 0\\ 0 & 0 & 1 & s\beta\\ 0 & 0 & -s\beta & 1\end{bmatrix}$,
$\Lambda_{(\mathrm{i}\xi,\beta)}=\gamma_{(\xi\beta)}\begin{bmatrix}1 & -\xi^2\beta^T\\ -\beta & \mathrm{I}_3+\mathrm{i}\xi\,\mathrm{A}_{(\beta)}\end{bmatrix}$,	$\Lambda_{(\xi,\beta)}=\gamma_{(\mathrm{i}s\beta)}\begin{bmatrix}1 & s^2\beta^T\\ -\beta & \mathrm{I}_3+s\,\mathrm{A}_{(\beta)}\end{bmatrix}$,
$g=-g_{00}\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & \xi^2 & 0 & 0\\ 0 & 0 & \xi^2 & 0\\ 0 & 0 & 0 & \xi^2\end{bmatrix}$,	$g=-g_{00}\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & -s^2 & 0 & 0\\ 0 & 0 & -s^2 & 0\\ 0 & 0 & 0 & -s^2\end{bmatrix}$,
$U=\frac{1}{\xi}\mathrm{c}\in\mathrm{R}_+$, $X\in\mathrm{RC}^3$.	U does not exist, $X\in\mathrm{R}^4$.

The results may be applied to any complex or real isotropic space of dimension four (spacetime), endowed with the corresponding metric, whose elements (four-vectors) have spatial part (vector) with *Euclidean metric*. We simply put

$$\mathrm{c}t\to x^0\ ;\ x\to x^1\ ;\ y\to x^2\ ;\ z\to x^3. \tag{180}$$

So, the β-factor is written as

$$\beta^i=\frac{\mathrm{d}x^i}{\mathrm{d}x^0}. \tag{181}$$

New spaces are produced from the initial space: (i) with derivation of the initial four-vector wrt an invariant quantity, such as ‘proper time’, or (ii) with multiplication of the initial four-vector with invariant quantity, such as ‘mass’ (see e.g. [4] p. 109). Moreover, there exist applications beyond physics as biometry, econometrics etc, producing suitable vectors and four-vectors.

The specific value $\omega_{\mathrm{I}}=\mathrm{i}$ ($\xi_{\mathrm{I}}=1$) gives the *Lorentzian-Einsteinian version of CILToCST* endowed with *Lorentz metric* (for $g_{\mathrm{I}ii}=1$), which produces the *Lorentzian Complex Relativity Theory*, which

- (i) is using the *Lorentzian-Einsteinian version of CILToCST* instead of *Lorentz transformation* that is used by ESR,
- (ii) is using complex Cartesian Coordinates creating a Generalized Euclidean Geometry,
- (iii) creates the *new group of CILToCST* with elements complex matrices, instead of the *Lorentz group* of ESR,
- (iv) can produce the *Lorenz Boost* of ESR [3] (p. 6),

(v) is successfully applied to mechanics and electromagnetism,
(vi) maintains the Classical Physical Laws and the formalism of ERT,
(vii) is in accordance with Quantum Mechanics,
(viii) gives results that are exactly the same as ERT, if only two observers are related,
(ix) gives different results than ERT, when more than two observers are related, and
(x) calculates with better accuracy the fine structure peeks of spectrum of atomic Hydrogen than ESR [8] (p. 4).

Finally, we can consider that the value of ω_{I} depends on the *cosmic time* (t_c). Thus, *Lorentz metric* is valid, only for 'events' near to (nowadays, Earth). This could be an explanation for the problem of *dark matter* and *dark energy*.

5 Improper isometric Linear Transformations in Spacetime endowed with Euclidean, or Lorentz, or generally Isotropic metric

In the derivation of proper closed isometric LT (ICLToCST) (↓↑), we have chosen positive *b* [the *lower sign* (↓) in (91)] and via (64) *f* is positive (↑), too. So, there have remained the following three (3) improper non-closed isometric LTs (which do not contain the *identity transformation*) [see also the *Lorentzian-Einsteinian version* [7] (pp. 1049-1050)]:

(i) *Space inversion non-closed isometric Linear Transformation* (↓↓) in *isotropic ST* and E^4 with corresponding matrices $\left(\det\Lambda = \det\tilde{R} = -1\right)$:

$$\Lambda_{(\beta)} = \gamma_{(\mathrm{i}\omega\beta)}\begin{bmatrix} 1 & \omega^2\beta^{\mathrm{T}} \\ \beta & -\mathrm{I}_3 - \omega\mathrm{A}_{(\beta)} \end{bmatrix} \quad ; \tag{182}$$

$$\tilde{R}_{(\omega\beta)} = \gamma_{(\mathrm{i}\omega\beta)}\begin{bmatrix} 1 & \omega\beta^{\mathrm{T}} \\ \omega\beta & -\mathrm{I}_3 - \omega\mathrm{A}_{(\beta)} \end{bmatrix} = \gamma_{(\mathrm{i}B)}\begin{bmatrix} 1 & B^{\mathrm{T}} \\ B & -\mathrm{I}_3 - \mathrm{A}_{(B)} \end{bmatrix} = \tilde{R}_{(B)} . \tag{183}$$

The respective *typical transformations* along the *x*-axis, have

$$\Lambda_{(x)(\beta)} = \gamma_{(\mathrm{i}\omega\beta)}\begin{bmatrix} 1 & \omega^2\beta & 0 & 0 \\ \beta & -1 & 0 & 0 \\ 0 & 0 & -1 & -\omega\beta \\ 0 & 0 & \omega\beta & -1 \end{bmatrix} \quad ; \tag{184}$$

$$\tilde{R}_{(x)(\omega\beta)} = \gamma_{(\mathrm{i}\omega\beta)}\begin{bmatrix} 1 & \omega\beta & 0 & 0 \\ \omega\beta & -1 & 0 & 0 \\ 0 & 0 & -1 & -\omega\beta \\ 0 & 0 & \omega\beta & -1 \end{bmatrix} = \gamma_{(\mathrm{i}B)}\begin{bmatrix} 1 & B & 0 & 0 \\ B & -1 & 0 & 0 \\ 0 & 0 & -1 & -B \\ 0 & 0 & B & -1 \end{bmatrix} = \tilde{R}_{(x)(B)} . \tag{185}$$

(ii) *Time inversion non-closed isometric Linear Transformation* (↑↑) in *isotropic ST* and E^4 with corresponding matrices $\left(\det\Lambda = \det\tilde{R} = -1\right)$:

$$\Lambda_{\mathrm{B}(\beta)} = \gamma_{(\mathrm{i}\,\omega\beta)} \begin{bmatrix} -1 & -\omega^2\beta^{\mathrm{T}} \\ -\beta & \mathrm{I}_3 + \omega\mathrm{A}_{(\beta)} \end{bmatrix} \; ; \tag{186}$$

$$\tilde{R}_{(\omega\beta)} = \gamma_{(\mathrm{i}\,\omega\beta)} \begin{bmatrix} -1 & -\omega\beta^{\mathrm{T}} \\ -\omega\beta & \mathrm{I}_3 + \omega\mathrm{A}_{(\beta)} \end{bmatrix} = \gamma_{(\mathrm{i}\,B)} \begin{bmatrix} -1 & -B^{\mathrm{T}} \\ -B & \mathrm{I}_3 + \mathrm{A}_{(B)} \end{bmatrix} = \tilde{R}_{(B)} . \tag{187}$$

The respective *typical transformations* along the *x*-axis, have

$$\Lambda_{(x)(\beta)} = \gamma_{(\mathrm{i}\,\omega\beta)} \begin{bmatrix} -1 & -\omega^2\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega\beta \\ 0 & 0 & -\omega\beta & 1 \end{bmatrix} \; ; \tag{188}$$

$$\tilde{R}_{(x)(\omega\beta)} = \gamma_{(\mathrm{i}\,\omega\beta)} \begin{bmatrix} -1 & -\omega\beta & 0 & 0 \\ -\omega\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & \omega\beta \\ 0 & 0 & -\omega\beta & 1 \end{bmatrix} = \gamma_{(\mathrm{i}\,B)} \begin{bmatrix} -1 & -B & 0 & 0 \\ -B & 1 & 0 & 0 \\ 0 & 0 & 1 & B \\ 0 & 0 & -B & 1 \end{bmatrix} = \tilde{R}_{(x)(B)} . \tag{189}$$

(iii) *Spacetime inversion non-closed isometric Linear Transformation* (↑↓) in *isotropic ST* and E^4 with corresponding matrices $\left(\det\Lambda = \det\tilde{R} = 1\right)$:

$$\Lambda_{(\beta)} = \gamma_{(\mathrm{i}\,\omega\beta)} \begin{bmatrix} -1 & -\omega^2\beta^{\mathrm{T}} \\ \beta & -\mathrm{I}_3 - \omega\mathrm{A}_{(\beta)} \end{bmatrix} \; ; \tag{190}$$

$$\tilde{R}_{(\omega\beta)} = \gamma_{(\mathrm{i}\,\omega\beta)} \begin{bmatrix} -1 & -\omega\beta^{\mathrm{T}} \\ \omega\beta & -\mathrm{I}_3 - \omega\mathrm{A}_{(\beta)} \end{bmatrix} = \gamma_{(\mathrm{i}\,B)} \begin{bmatrix} -1 & -B^{\mathrm{T}} \\ B & -\mathrm{I}_3 - \mathrm{A}_{(B)} \end{bmatrix} = \tilde{R}_{(B)} . \tag{191}$$

The respective *typical transformations* along the *x*-axis, have

$$\Lambda_{(x)(\beta)} = \gamma_{(\mathrm{i}\,\omega\beta)} \begin{bmatrix} -1 & -\omega^2\beta & 0 & 0 \\ \beta & -1 & 0 & 0 \\ 0 & 0 & -1 & -\omega\beta \\ 0 & 0 & \omega\beta & -1 \end{bmatrix} \; ; \tag{192}$$

$$\tilde{R}_{(x)(\omega\beta)} = \gamma_{(\mathrm{i}\,\omega\beta)} \begin{bmatrix} -1 & -\omega\beta & 0 & 0 \\ \omega\beta & -1 & 0 & 0 \\ 0 & 0 & -1 & -\omega\beta \\ 0 & 0 & \omega\beta & -1 \end{bmatrix} = \gamma_{(\mathrm{i}\,B)} \begin{bmatrix} -1 & -B & 0 & 0 \\ B & -1 & 0 & 0 \\ 0 & 0 & -1 & -B \\ 0 & 0 & B & -1 \end{bmatrix} = \tilde{R}_{(x)(B)} . \tag{193}$$

These matrices are exactly the opposite of the corresponding *proper ICLToCST*.

The above can be compared to the case of *Lorentz Boost* [4] (pp. 30-31), [7] (pp. 1050-1052), where we have:

(a) *Space inversion Lorentz Boost* in M^4 and E^4 with corresponding matrices $\left(\det\Lambda_{\mathrm{L}} = \det\tilde{R}_{\mathrm{L}} = -1\right)$:

$$\Lambda_{\mathrm{L}(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & \gamma_{(\beta)}\beta^{\mathrm{T}} \\ -\gamma_{(\beta)}\beta & -\mathrm{I}_3 - \dfrac{\gamma_{(\beta)}-1}{\beta^{\mathrm{T}}\beta}\beta\beta^{\mathrm{T}} \end{bmatrix}; \tag{194}$$

$$\tilde{R}_{\mathrm{L}(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -\mathrm{i}\gamma_{(\beta)}\beta^{T} \\ -\mathrm{i}\gamma_{(\beta)}\beta & -\mathrm{I}_3 - \dfrac{\gamma_{(\beta)}-1}{\beta^{T}\beta}\beta\beta^{T} \end{bmatrix} = \begin{bmatrix} \gamma_{(\mathrm{i}B)} & -\gamma_{(\mathrm{i}B)}B^{T} \\ -\gamma_{(\mathrm{i}B)}B & -\mathrm{I}_3 - \dfrac{\gamma_{(\mathrm{i}B)}-1}{B^{T}B}BB^{T} \end{bmatrix} = \tilde{R}_{\mathrm{L}(B)}. \tag{195}$$

The respective *typical transformations* along the *x*-axis, have

$$\Lambda_{\mathrm{L}(x)(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & \gamma_{(\beta)}\beta & 0 & 0 \\ -\gamma_{(\beta)}\beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}; \tag{196}$$

$$\tilde{R}_{\mathrm{L}(x)(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -\mathrm{i}\gamma_{(\beta)}\beta & 0 & 0 \\ -\mathrm{i}\gamma_{(\beta)}\beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \gamma_{(\mathrm{i}B)} & -\gamma_{(\mathrm{i}B)}B & 0 & 0 \\ -\gamma_{(\mathrm{i}B)}B & -\gamma_{(\mathrm{i}B)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \tilde{R}_{\mathrm{L}(x)(B)}. \tag{197}$$

(b) *Time inversion Lorentz Boost* in M^4 and E^4 with corresponding matrices $\left(\det\Lambda_{\mathrm{L}} = \det\tilde{R}_{\mathrm{L}} = 1\right)$:

$$\Lambda_{\mathrm{L}(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & \gamma_{(\beta)}\beta^{T} \\ \gamma_{(\beta)}\beta & \mathrm{I}_3 - \dfrac{\gamma_{(\beta)}+1}{\beta^{T}\beta}\beta\beta^{T} \end{bmatrix}; \tag{198}$$

$$\tilde{R}_{\mathrm{L}(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & -\mathrm{i}\gamma_{(\beta)}\beta^{\mathrm{T}} \\ \mathrm{i}\gamma_{(\beta)}\beta & \mathrm{I}_3 - \dfrac{\gamma_{(\beta)}+1}{\beta^{\mathrm{T}}\beta}\beta\beta^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} -\gamma_{(\mathrm{i}B)} & -\gamma_{(\mathrm{i}B)}B^{\mathrm{T}} \\ \gamma_{(\mathrm{i}B)}B & \mathrm{I}_3 - \dfrac{\gamma_{(\mathrm{i}B)}+1}{B^{\mathrm{T}}B}BB^{\mathrm{T}} \end{bmatrix} = \tilde{R}_{\mathrm{L}(B)}. \tag{199}$$

The respective *typical transformations* along the *x*-axis, have

$$\Lambda_{\mathrm{L}(x)(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & \gamma_{(\beta)}\beta & 0 & 0 \\ \gamma_{(\beta)}\beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \tag{200}$$

$$\tilde{R}_{\mathrm{L}(x)(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & -\mathrm{i}\gamma_{(\beta)}\beta & 0 & 0 \\ \mathrm{i}\gamma_{(\beta)}\beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\gamma_{(\mathrm{i}B)} & -\gamma_{(\mathrm{i}B)}B & 0 & 0 \\ \gamma_{(\mathrm{i}B)}B & -\gamma_{(\mathrm{i}B)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \tilde{R}_{\mathrm{L}(x)(B)}. \tag{201}$$

(c) *Spacetime inversion Lorentz Boost* in M^4 and E^4 with corresponding matrices $\left(\det\Lambda_{\mathrm{L}} = \det\tilde{R}_{\mathrm{L}} = -1\right)$:

$$\Lambda_{\mathrm{L}(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & -\gamma_{(\beta)}\beta^{\mathrm{T}} \\ \gamma_{(\beta)}\beta & -\mathrm{I}_3 + \dfrac{\gamma_{(\beta)}+1}{\beta^{\mathrm{T}}\beta}\beta\beta^{\mathrm{T}} \end{bmatrix}; \tag{202}$$

$$\tilde{R}_{\mathrm{L}(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & \mathrm{i}\gamma_{(\beta)}\beta^{T} \\ \mathrm{i}\gamma_{(\beta)}\beta & -\mathrm{I}_3 + \dfrac{\gamma_{(\beta)}+1}{\beta^{\mathrm{T}}\beta}\beta\beta^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} -\gamma_{(\mathrm{i}B)} & \mathrm{i}\gamma_{(\mathrm{i}B)}B^{T} \\ \mathrm{i}\gamma_{(\mathrm{i}B)}B & -\mathrm{I}_3 + \dfrac{\gamma_{(\mathrm{i}B)}+1}{B^{\mathrm{T}}B}BB^{\mathrm{T}} \end{bmatrix} = \tilde{R}_{\mathrm{L}(B)}. \tag{203}$$

The respective *typical transformations* along the *x*-axis, have

$$\Lambda_{\mathrm{L}(x)(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & -\gamma_{(\beta)}\beta & 0 & 0 \\ \gamma_{(\beta)}\beta & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}; \tag{204}$$

$$\tilde{R}_{\mathrm{L}(x)(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & \mathrm{i}\gamma_{(\beta)}\beta & 0 & 0 \\ \mathrm{i}\gamma_{(\beta)}\beta & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -\gamma_{(\mathrm{i}B)} & \gamma_{(\mathrm{i}B)}B & 0 & 0 \\ \gamma_{(\mathrm{i}B)}B & \gamma_{(\mathrm{i}B)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \tilde{R}_{\mathrm{L}(x)(B)}. \tag{205}$$

5 Conclusions

In a 3D complex ‘space’ endowed with *Euclidean metric,* we consider one frame O*xyz*, where a ‘position’ vector has real *Cartesian Coordinates*. Another real independent variable (‘time’) and the aforementioned coordinates produce a real four-vector. There exist *two cases* of closed linear ‘spacetime’ transformation of this real four-vector: *one* with the ‘time’ depending on the position, where the ‘event’ happens and *another* with the ‘time’ being independent from the position. The *first case* can have real Invariant ‘Speed’ (*U*), in contrast to the *second case*, which has only infinite *U*.

Moreover, for transformation having *isometry*, the *first case* transformation matrix is totally calculated and contains a parameter ω, with $\omega^2 = g_{ii}/g_{00}$ (the ratio of coefficients of ‘spacetime’ metric) in addition to the ‘velocity’ of the frame $\mathrm{O}'x'y'z'$ wrt O*xyz*. The *second case* is turned to *Galilean Transformation* (GT). The assumption that $\omega \to 0$ in the *first case* yields GT. So, in isometry, the *second case is embedded to the first case* transformation. Besides the *first case* is divided to *two types*: *one type*, where ‘time’ and ‘space’ have ‘spacetime’ metric coefficients with different signs [signature of spacetime: (-+++)], which leads to complex 3D ‘space’ with real *U*. The *second type*, where ‘time’ and ‘space’ have metric coefficients with the same sign [signature of spacetime: (----)], leads to real 3D ‘space’ without *U*. *Time remains real*, in both cases.

If the *metric is independent from the ‘position’* of the ‘event’ in ‘spacetime’, it is $\omega = \omega_{\mathrm{I}}$=constant and we have the case of ‘Special Relativity’

('SR') and the transformation can be applied globally, relating 'Inertial Observers / Frames' ('IOs/Fs'). Thus, infinite number of 'SRs' are produced (each one with the corresponding metric), all of them keeping Einsteinian SR-formalism. In the case that *metric depends on the 'position'* of the 'event' in 'spacetime', we have the 'General Relativity' ('GR') and the transformation may be applied locally, relating 'accelerated observers / frames'. Thus, infinite number of 'GRs' are produced (each one with the corresponding metric of IOs' spacetime), all of them keeping Einsteinian GR-formalism. Of course, vanishing 'acceleration' leads to the corresponding 'SR'.

This new modeling of study allows studying *Einsteinian Relativity Theory*, *Newtonian Physics* (NPs), or any other Theory of Physics that is in accordance with closed Linear Spacetime Transformations simultaneously. This is achieved, because the coefficients of spacetime metric are contained in the transformation matrix. Besides, NPs is obtained, not only by the low velocity limit, but also by the zero limit of the space coefficient of spacetime metric ($g_{ii}\to 0$), or equivalently $\omega\to 0$. Finally, we can consider that the value of ω_{I} depends on the *cosmic time* (t_{C}) and *Lorentz metric* is valid only for 'events' near to the (nowadays, Earth). This could be used for the explanation of *dark matter* and *dark energy*.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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