Ratio Mathematica Volume 47, 2023

Transit in Corona product of graphs

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Abstract

For a graph $G(V, E)$, the transit of a vertex v is defined as the sum of the lengths of all geodesics with v as an internal vertex. This paper deals with the transit of vertices in Corona product of graphs. We obtain expressions for transit of an arbitrary vertex in the Corona product $G_1 \circ G_2$ of G_1 and G_2. We also consider the cases where G_1 is a particular graph class. The expressions for transit of vertices in $G_1 \circ G_2$, with G_1 as path P_n, a cycle C_n, a star S_n, a complete graph K_n and a complete bipartite graph $K_{m,n}$ is established.

Keywords: Geodesic, Transit Index, Transit Identical, Corona product

2020 AMS subject classifications: 05C10, 05C12 [1]

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[1]Received on September 24, 2022. Accepted on June 18, 2023. Published on June 30, 2023. DOI: 10.23755/rm.v39i0.860. ISSN: 1592-7415. eISSN: 2282-8214. ©The Authors. This paper is published under the CC-BY licence agreement.

1 Introduction

Topological indices and centrality measures are graph invariant. Numerous studies have been carried out in these areas. The first notable topological index was the wiener index, named after Harry Wiener, a pioneer in chemical graph theory. It is defined as the sum of the lengths of the shortest paths between all pairs of vertices in the chemical graph representing the non-hydrogen atoms in the molecule. Wiener made fundamental contributions to the study of topological indices and established a correlation between the Wiener index and boiling points (hence viscosity and surface tension) of the paraffin. He could establish relationships with many chemical properties of alkanes with the wiener index.

Centrality measures are a vital tool for understanding graphs. Each measure has its own definition of importance. Some are based on the degree of a vertex while others take the closeness to other vertices as the score of significance.

In the paper [Shimbel and Alfonso], they introduced the concept of the stress of a vertex. It is the number of shortest paths on which a vertex lies. This was further modified to produce measures of centrality. It found applications in social networking, for analyzing communication dynamics.

Keeping in mind the above two concepts, we introduced a new index, called the transit index of a graph. It considers the distances in the graph as well as the degree of vertices. In computing the stress of a vertex, we only take into account the number of shortest paths through it; the length of the paths is not considered. Be it in data transmission or in the measure of closeness, the length of the paths also matters. Hence, in the computation of transit we account for the number of shortest paths as well as their length.

Graph products are binary operations. Two graphs G_1 and G_2 are combined to produce a new graph H. In this paper, we study the transit of vertices in the Corona product of graphs. Information on individual graphs is used to compute the transit of vertices in their Corona products. Thus the transit of vertices in large graphs and networks, which can be viewed as Corona products of simple graphs, can be computed more efficiently.

2 Preliminaries

In this section, we come across certain definitions and terminologies employed in developing results in the latter sections. Throughout this paper we only consider simple connected and finite graphs.

Definition 2.1 (K.M.Reshmi and Pilakkat. Raji [2020]). *Let $v \in V$. Then the transit of v denoted by $T(v)$ is "the sum of the lengths of all shortest paths with v*

as an internal vertex" and the transit index of G denoted by $TI(G)$ is

$$TI(G) = \sum_{v \in V} T(v)$$

Lemma 2.2 (K.M.Reshmi and Pilakkat. Raji [2020]). *$T(v) = 0$ iff $\langle N[v] \rangle$ is a clique.*

Theorem 2.3 (K.M.Reshmi and Pilakkat. Raji [2020]). *For a path P_n, Transit index is*

$$TI(P_n) = \frac{n(n+1)(n^2-3n+2)}{12}$$

Theorem 2.4. *For a cycle, the transit of any vertex v is, $T(v) = \frac{(n^2-4)n}{24}$ and*

i) $TI(C_n) = \frac{n^2(n^2-4)}{24}$,n even.

ii) $TI(C_n) = \frac{n(n^2-1)(n-3)}{24}$, n odd

Definition 2.5. *Two vertices of a graph are called transit identical if the shortest paths passing through it are same in number and length.*

We use the following terminologies.
The order of a graph G, denoted by $|G|$ is the number of vertices in $V(G)$. The distance between two vertices $u, v \in V$ is the length of any shortest $u - v$ path in G. A shortest path from u to v is also called a $u - v$ geodesic. The number of shortest $u - v$ paths is denoted by $\sigma(u, v)$ and the number of shortest $u - v$ path with 'a' as an internal vertex is denoted by $\sigma(u, v/a)$. It can be noted that a vertex 'a' lies on a shortest $u - v$ path iff $d(u, v) = d(u, a) + d(a, v)$. The number of shortest $u - v$ path with 'a' as an internal vertex can be computed as $\sigma(u, v/a) = \sigma(u, a) \times \sigma(a, v)$. Number of shortest paths in G with 'a' as an internal vertex is denoted by $\sigma_G(a)$. Clearly $\sigma_G(a) = \sum_{(u,v)} \sigma(u, v/a)$

3 Transit of vertices in Corona Product of graphs

3.1 Corona Product of Graphs

Definition 3.1. *(Frucht and Harary [1970]) Let G_1 and G_2 be two graphs. The corona product $G_1 \circ G_2$, is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2; and by joining each vertex of the i-th copy of G_2 to the i-th vertex of G_1, where $1 \leq i \leq |V(G_1)|$*

Whenever we consider $G_1 \circ G_2$, we use the following notations.
1. G_2^i the ith copy of G_2 in $G_1 \circ G_2$
2. $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}, |E(G_1)| = m_1$
3. $V(G_2^i) = \{v_1^i, v_2^i, \ldots, v_{n_2}^i\}, |E(G_2^i)| = m_2, \forall i$

Lemma 3.2. *(Agnes [2015]) Let G_1 and G_2 be two arbitrary graphs. Then,*

- $d_{G_1 \circ G_2}(u_i, u_p) = d_{G_1}(u_i, u_p), 0 \leq i, p \leq n_1$
- $d_{G_1 \circ G_2}(u_i, v_q^p) = d_{G_1}(u_i, u_p) + 1, 0 \leq i, p \leq n_1, 0 \leq q \leq n_2$
- $d_{G_1 \circ G_2}(v_j^i, v_q^p) = \begin{cases} d_{G_1}(u_i, u_p) + 2, & i \neq p \\ 1, & if \;\; i = p, \;\; v_j v_q \in E(G_2) \\ 2, & if \;\; i = p, \;\; v_j v_q \notin E(G_2) \end{cases}$

Proposition 3.3. *For any two graphs G_1 and G_2,*

1. *$T_{G_2}(a) = 0$ iff $T_{G_1 \circ G_2}(a) = 0$, $a \in G_2$*
2. *$T_{G_2}(a) = T_{G_1 \circ G_2}(a)$, for $a \in G_2$ iff every shortest path in G_2 with 'a' as an internal vertex is of length 2.*

Proof. Note that as the result 2 is obvious, we prove only 1. 1) $T_{G_2}(a) = 0 \Longleftrightarrow \langle N_{G_2}[a] \rangle$ is a clique. $\Longleftrightarrow \langle N_{G_1 \circ G_2}[a] \rangle$ is a clique$\Longleftrightarrow T_{G_1 \circ G_2}(a) = 0$. □

Proposition 3.4. *Let G_1 and G_2 be arbitrary graphs,*
1)For any u_p in G_1,
$\sigma_{G_1 \circ G_2}(u_p) = (n_2 + 1)\Big[(n_2 + 1)\sigma_{G_1}(u_p) + n_2 \sum_{p \neq k=1}^{n_1} \sigma_{G_1}(u_p, u_k)\Big]$
2) $\sigma_{G_1 \circ G_2}(v_k^i)$ = Number of geodesic of length 2 in G_2 with v_k as an internal vertex.

Proof. 1)Let u_p be any vertex of G_1. Every geodesic in G_1 with u_p as an internal vertex will be counted in $\sigma_{G_1 \circ G_2}(u_p)$.
For $k \neq l$, geodesics connecting $G_2^k \cup \{u_k\}$ to $G_2^l \cup \{u_l\}$ will have $u_l - u_k$ geodesic as its part. Let P_1 be one of the $u_l - u_k$ geodesic with u_p as an internal vertex. Then P_1 will be part of a geodesic connecting vertices of $G_2^k \cup \{u_k\}$ to vertices of $G_2^l \cup \{u_l\}$. There will be
- n_2^2 geodesics connecting G_2^k to G_2^l,
- n_2 geodesics connecting G_2^k to u_l and

- n_2 geodesics connecting G_2^l to u_k, with P_1 as its part. Hence for the pair of vertices (u_k, u_l), there will be $({n_2}^2 + 2n_2 + 1)\sigma_{G_1}(u_p)$ geodesics with u_p as an internal vertex.
The geodesics connecting vertices of G_2^p to other vertices of $G_1 \circ G_2$ will have $u_p - u_k$ geodesic as a part for some k. If P_2 is one of the $u_p - u_k$ geodesic, it will be part of
- n_2^2 geodesics connecting G_2^k to G_2^p and
- n_2 geodesics connecting G_2^p to u_k. Hence for every $u_p - u_k$ geodesic in G_1 there will be $\sigma_{G_1}(u_p, u_k)[n_2(n_2+1)]$ geodesics in $G_1 \circ G_2$ with u_p as an internal vertex. Considering every pair $u_k - u_p$ the result follows.
2) Since every vertex of G_2^i are joined to u_i, the maximum distance between vertices of G_2^i is 2. Hence the proof. □

Next we find an expression for the transit of a vertex, u_p in $G_1 \circ G_2$, where G_1 and G_2 are arbitrary. Let (u_k, u_l) be a pair of vertices in G_1 such that $u_k - u_l$ geodesic has u_p as an internal vertex. Let $T_{kl}(u_p)$ denote the contribution to transit of u_p, due to geodesic connecting vertices of $G_2^k \cup \{u_k\}$ to $G_2^l \cup \{u_l\}$. Also we denote the contribution of vertices in G_2^p to $T(u_p)$ by $T_p(u_p)$.

Lemma 3.5. *For arbitrary graphs G_1 and G_2,*

$$T_{kl}(u_p) = \sigma_{G_1}(u_k, u_l/u_p)\Big[(n_2+1)^2 d(u_k, u_l) + 2n_2(n_2+1)\Big]$$

Proof. Table 1 gives the length and number of geodesics through u_p

Vertices connected	Length	Number
G_2^k to G_2^l	$2 + d(u_k, u_l)$	$n_2^2\sigma(u_k, u_l/u_p)$
u_k to G_2^l	$1 + d(u_k, u_l)$	$n_2\sigma(u_k, u_l/u_p)$
G_2^k to u_l	$1 + d(u_k, u_l)$	$n_2\sigma(u_k, u_l/u_p)$
u_k to u_l	$d(u_k, u_l)$	$\sigma(u_k, u_l/u_p)$

Table 1: Table detailing geodesics through u_p

The result follows. □

Lemma 3.6. $T_p(u_p) = \sum_{p \neq k=1}^{n_1} \Big[\sigma_{G_1}(u_p, u_k)\big[n_2(n_2+1)d(u_p, u_k) + n_2(1+2n_2)\big]\Big] + 2\left[\binom{n_2}{2} - m_2\right]$

Proof. Table 2 gives the contribution of geodesics through u_p to $T_p(u_p)$. Considering every vertex $u_k, k \neq p$, the result follows. □

Theorem 3.7. *1) $T(u_p) = T_p(u_p) + \sum_{kl} T_{kl}(u_p)$*
2) $T(v_i^p) = 2\times$ number of geodesics in G_2 of length 2 through v_i.

Vertices connected	Length	Number
u_k to G_2^p	$1 + d(u_k, u_p)$	$n_2\sigma(u_k, u_p)$
G_2^k to G_2^p	$2 + d(u_k, u_p)$	$n_2^2\sigma(u_k, u_p)$
G_2^p to G_2^p	2	$\binom{n_2}{2} - m_2$

Table 2: Geodesics through u_p

Proof. 1) Geodesics through u_p are either considered in $T_p(u_p)$ or in $T_{kl}(u_p)$. Hence the result is evident.
2) Follows from Proposition 3.4. □

In the remaining sections we consider G_2 as arbitrary, while G_1 is replaced by various graph classes like $P_n, C_n, K_n, K_{m,n}$ and S_{m+1}

3.2 Path Graphs

Let P_n be the path graph with vertices $1, 2, \ldots, n$. We give an expression for transit of k using Theorem 3.7 in $P_n \circ G_2$.

Theorem 3.8.

$$T(k) = \frac{(k-1)(n_2+1)(n-k)}{2}\Big[(n_2+1)(n+1) + 4n_2\Big] +$$

$$n_2(n_2+1)\Big[\frac{(k-1)k}{2} + \frac{(n-k)(n-k+1)}{2}\Big] + n_2(2n_2+1)(n-1) + 2\Big[\binom{n_2}{2} - m_2\Big]$$

Proof. Let $1 \leq l < k < m \leq n$. Since G_1 is a path, we have $\sigma_{G_1}(l, m/k) = 1$. Hence $T_{lm}(k) = (n_2+1)^2(m-l) + 2n_2(n_2+1)$

$$\therefore \sum_{l,m} T_{lm}(k) = (n_2+1)^2 \sum_{l=1}^{k-1} \sum_{m=k-1}^{n} (m-l) + \sum_{l=1}^{k-1} \sum_{m=k-1}^{n} 2n_2(n_2+1)$$

$$= (n_2+1)^2 T_{G_1}(k) + (k-1)(n-k-1)2n_2(n_2+1)$$

$$= (n_2+1)^2 \frac{(n+1)(k-1)(n-k)}{2} + (k-1)(n-k-1)2n_2(n_2+1)$$

$$= \frac{(k-1)(n_2+1)(n-k)}{2}\Big[(n_2+1)(n+1) + 4n_2\Big]$$

In a similar manner we compute $T_k(k)$

$$T_k(k) = \sum_{k\neq i=1}^{n} \Big[(d(k,i)+1)n_2 + (d(k,i)+2)n_2^2\Big]$$
$$+2\Big[\binom{n_2}{2} - m_2\Big]$$
$$= (n_2+n_2^2)\left[\frac{(k-1)k}{2} + \frac{(n-k)(n-k+1)}{2}\right] + n_2(2n_2+1)(n-1)$$
$$+2\Big[\binom{n_2}{2} - m_2\Big]$$

The result follows. □

In the following examples we compute transit for the vertices in various corona product of P_n. From Theorem 3.8, we have

$$T(k) = \frac{(k-1)(n_2+1)(n-k)}{2}\Big[(n_2+1)(n+1)+4n_2\Big]$$

$$+n_2(n_2+1)\Big[\frac{(k-1)k}{2}+\frac{(n-k)(n-k+1)}{2}\Big]+n_2(2n_2+1)(n-1)+2\Big[\binom{n_2}{2}-m_2\Big]$$

$= T_1 + T_2$, say where $T_2 = 2\Big[\binom{n_2}{2} - m_2\Big]$

Examples

1. $G_2 = P_m$. Here $n_2 = m, m_2 = m-1$. Hence $T(k) = T_1 + (m-1)(m-2)$. For pendant vertices of P_m^i, the transit is 0 and 2 for others, $\forall i$.

2. $G_2 = C_m$. Here $n_2 = m_2 = m$. For every vertex in C_m^i, there exist only one geodesic of length 2 through it.
   Here $T(k) = T_1 + m(m-3)$ and $T(v_k^i) = 2$, $\forall k, i$.

3. $G_2 = K_m$. Then $n_2 = m, m_2 = \binom{m}{2}$.
   Thus, $T(k) = T_1$ and $T(v_k^i) = 0$, $\forall k, i$.

4. $G_2 = S_m$. Here $n_2 = m, m_2 = m-1$. Hence $T(k) = T_1 + (m-1)(m-2)$. $T(v_k^i) = 0$, for pendant vertices and for central vertex of $S-m$, $T(v_k^i) = (m-1)(m-2)$

5. $G_2 = K_{l_1,l_2}$. $n_2 = m = l_1 + l_2$ and $m = l_1 l_2$
   $T(k) = T_1 + (m-1)m - 2l_1l_2$ and $T(v_k^i) = T_{K_{l_1,l_2}}(v_k)$

3.3 Cycle

In this section we consider G_1 to be a cycle. We have already seen that transit of vertices in cycles with order $2n$ and $2n+1$ are the same. Hence we consider $G_1 = C_{2n_1+1}$. We represent the vertices by $0, 1, \ldots, 2n_1$. Also every vertex in the cycle being transit identical, it is enough we compute the transit for n_1.

Theorem 3.9. *If a is any vertex of the cycle C_{2n} or C_{2n+1}, its transit in the corona product $C_{2n} \circ G_2$ or $C_{2n+1} \circ G_2$ is given by* $T(a) = \frac{(n_2+1)(n_1-1)n_1}{3}[n_2n_1 + 4n_2 + n_1 + 1] + n_2n_1\big[(n_2+1)(n_1+1) + 2(1+2n_2)\big] + 2\left[\binom{n_2}{2} - m_2\right]$

Proof. For any k, l we know that $\sigma_{G_1}(k, l/n_1) = 1$.

$$T_{k,l}(n_1) = \Big[(n_2+1)^2 d(k,l) + 2n_2(n_2+1)\Big]$$

$$\begin{aligned}
\therefore \sum_{k,l} T_{k,l}(n_1) &= (n_2+1)^2 \sum_{k,l} d(k,l) + \frac{n_1(n_1-1)}{2} 2n_2(n_2+1) \\
&= (n_2+1)^2 T_{G_1}(n_1) + \frac{n_1(n_1-1)}{2} 2n_2(n_2+1) \\
&= (n_2+1)^2 \frac{(n^2-1)n}{24} + \frac{n_1(n_1-1)}{2} 2n_2(n_2+1) \\
&= \frac{(n_2+1)(n_1-1)n_1}{3}[n_2n_1 + 4n_2 + n_1 + 1]
\end{aligned}$$

Next we compute $T_{n_1}(n_1)$

$$\begin{aligned}
T_{n_1}(n_1) = \sum_{n_1 \neq i=0}^{2n_1} & \Big[n_2(n_2+1)d(n,i) + n_2(1+2n_2)\Big] \\
& +2\left[\binom{n_2}{2} - m_2\right], \sigma_{G_1}(n_1, i) \text{ being } 1 \\
= n_2n_1\big[(n_2+1)(n_1+1) + 2(1+2n_2)\big] + 2\left[\binom{n_2}{2} - m_2\right]
\end{aligned}$$

And $T_{G_1 \circ G_2}(n_1) = \sum_{k,l} T_{k,l}(n_1) + T_{n_1}(n_1)$. Hence the proof. $\square$

3.4 Star

Let $G_1 = S_{n+1}$. In a star there are n pendant vertices and one central vertex. All pendant vertices are transit identical. Hence we need to compute transit of one of the pendant vertex and the central vertex in $S_{n+1} \circ G_2$. Let us name the vertices as $1, 2, \ldots, n+1$, where $n+1$ is the central vertex.

Theorem 3.10. *In $S_{n+1} \circ G_2$, $T(n+1) = n\big[(n-1)(n_2+1)(2n_2+1) + n_2(3n_2+2)\big] + 2\left[\binom{n_2}{2} - m_2\right]$ and $T(i) = n_2\big[(n_2+1)(2n-1) + n(2n_2+1)\big] + 2\left[\binom{n_2}{2} - m_2\right], i \neq n+1$*

Proof. Consider $n+1$. We have $\sigma(k, l/(n+1)) = 1$ and $d(k,l) = 2$

$$\text{Thus } T_{k,l}(n+1) = 2(n_2+1)(2n_2+1) \tag{1}$$

$$\therefore \sum_{k,l} T_{k,l}(n+1) = \binom{n}{2} 2(n_2+1)(2n_2+1) \tag{2}$$

$$= n(n-1)(n_2+1)(2n_2+1) \tag{3}$$

While computing $T_{n+1}(n+1)$, we see that $\sigma_{S_{n+1}}(n+1, i) = 1$ and $d(n+1,i) = 1, \forall i$. Thus we get $T_{n+1}(n+1) = nn_2(3n_2+2) + 2\left[\binom{n_2}{2} - m_2\right]$, which completes the computation for $T(n+1)$

Now consider the vertex $i \neq n+1$. It can easily be verified that $\sigma(k, l/i) = 0, \forall k, l$. Hence $\sum_{k,l} T_{k,l}(i) = 0$. for a fixed i, $\sigma(i,k) = 1 \forall k$ and $d(i, n+1) = 1$ and $d(i,k) = 2, k \neq n+1$

$\therefore T_i(i) = n_2\big[4nn_2 + 3n - n_2 - 1\big] + 2\left[\binom{n_2}{2} - m_2\right]$. Hence the proof. □

3.5 Complete graph and Complete bipartite graph

Theorem 3.11. *In the corona product $K_n \circ G_2$, the transit of any vertex of K_n is $(n-1)n_2(3n_2+2) + 2\left[\binom{n_2}{2} - m_2\right]$*

Proof. Since every vertex of K_n is transit identical, we consider one of them. Let u_i be any vertex of K_n.

$\sigma(u_k, u_l/u_i) = 0 \Longrightarrow \sum T_{k,l}(u_i) = 0$

Again $\sigma(u_i, u_k) = 1, \forall k \neq i$ and $d(u_i, u_k) = 1$

$\therefore T_i(u_i) = \sum_{k \neq i} \big[n_2(n_2+1) + n_2(2n_2+1)\big] + 2\left[\binom{n_2}{2} - m_2\right]$. Hence the result. □

Next we consider a complete bipartite graph K_{l_1,l_2} with bipartition V_1, V_2. Let $V_1 = a_1, a_2, \ldots, a_{l_1}$ and $V_2 = b_1, b_2, \ldots, b_{l_2}$. Then all a_i are transit identical. Similarly all b_i are also transit identical. Computation of $T(a_i)$ and $T(b_i)$ are similar. Hence we compute $T(a_i)$ only.

Theorem 3.12. *In $K_{l_1,l_2} \circ G_2$, the transit of a_i, $T(a_i) = \binom{l_2}{2} 2(n_2+1)(2n_2+1) + l_2 n_2(4n_2+3)(l_1-1) + l_2 n_2(3n_2+2) + 2\left[\binom{n_2}{2} - m_2\right]$*

Proof. The shortest path in K_{l_1,l_2} through a_i are those connecting vertices of V_2. $\therefore T_{k,l}(a_i) = \sigma_{K_{l_1,l_2}}(b_k, b_l/a_i)\big[(n_2+1)^2 d(u_k,u_l) + 2n_2(n_2+1)\big]$. Thus $\sum_{k,l} T_{k,l}(a_i) = \binom{l_2}{2} 2(n_2+1)(2n_2+1)$.

While computing $T_i(a_i)$, we see that vertices in V_1 and V_2 behaves differently. Hence we split the summation as follows.

$$T_i(a_i) = \sum_{a_j} \sigma(a_i, a_j)\Big[n_2(n_2+1)d(a_i, a_j) + n_2(2n_2+1)\Big]$$

$$+ \sum_{b_j} \sigma(a_i, b_j)\Big[n_2(n_2+1)d(a_i, b_j) + n_2(2n_2+1)\Big] + 2\left[\binom{n_2}{2} - m_2\right]$$

$$= l_2 n_2(4n_2+3)(l_1-1) + l_2 n_2(3n_2+2) + 2\left[\binom{n_2}{2} - m_2\right]$$

□

4 Conclusion

In this paper, we first considered arbitrary graphs G_1 and G_2. We could give an expression for the transit of vertices in their corona product. This result was applied to compute the transit of vertices in $G_1/circG_2$, where G_1 refers to a particular graph. It should be noted that we could express the transit of vertices in $G_1 \circ G_2$ in terms of individual graph parameters. Thus, the computation of transit in huge networks, which are Corona products, is now much easier.

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