Minimal Reinhard Zumkeller divisor cordial graphs

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Abstract

In this paper, the notion of minimal Reinhard Zumkeller divisor cordial labeling has been introduced. Let $G = (V, E)$ be a simple graph and $\gamma : V(G) \rightarrow minimum \; \{2^i \times 3, 2^{j+1} \times 5, 2^{k+1} \times 7, 2^l \times 3 \times 5, 2^m \times 3 \times 7 \; where \; i, j, k, l, m \geq 1\}$ be an injection such that the sum of the cardinality of exponent of $\gamma(V(G))$ should be equal to the order of the graph G. For each edge uv, assign the label 1 if $\gamma(u)|\gamma(v)$ or $\gamma(v)|\gamma(u)$ where $\gamma(u)$ and $\gamma(v)$ are Zumkeller numbers and the label 0 if $\gamma(u) \nmid \gamma(v)$ and also if $|e\gamma^|(0) - e\gamma^|(1)| \leq 1$ then γ is called minimal Reinhard Zumkeller divisor cordial labeling. This paper elucidates how the Zumkeller number, which is the generalization of the perfect number, goes along with the divisibility concept of the number theory and the cordial labeling technique. It also probes the existence of minimal Reinhard Zumkeller divisor cordial labeling of path, cycle, star $K_{1,s}$, complete bipartite, complete graph K_n for $n < 17$, tadpole graph $T_{n,k}$ for all values of n and k.
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1 Introduction

Graphs regarded here are finite, undirected and simple. The symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G. Most of the graph labeling methods trace their origin to the one introduced by Rosa [1967]. A graph labeling is an assignment of integer to the vertices or edges or both subject to certain conditions. Labelled graph has many branch out applications such as coding theory, missile guidance, X-ray, crystallography analysis, communication network addressing systems, astronomy, radar, circuit design, database management etc., The concept of cordial labeling was introduced by Cahit [1987]. Varatharajan et al. [2011] introduced divisor cordial labeling. If the sum of all the proper positive divisors of a positive integer is equal to the number, then the number is called perfect number. Generalizing the concept of perfect numbers R.H.Zumkeller defined a new type of number as a Zumkeller number. Peng and Rao [2013] established several results and conjectures on Zumkeller numbers. The notion of Zumkeller labeling of some cycle related graphs was investigated by Balamurugan et al. [2014]. Murali et al. [2017] proved results about Zumkeller cordial labeling of cycle related graphs. Shahbaz and Mahmood [2020] proved that Zumkeller number is either a super totient or a hyper totient number. Graph labeling has a potent communication between the number theory and graph network. The idea behind this work fosters us to develop a graph labeling technique called minimal Reinhard Zumkeller divisor cordial labeling by pooling the divisor cordial graph labeling technique and characteristics of Zumkeller number which is one of the engrossing parts of the number theory. Because of the existence of diverse γ vertex labeling design for some graph structure, minimal condition is emphasized. As the work has been focused on minimization condition, Zumkeller number chosen for the concept depicted herein is a sequence of least even Zumkeller numbers. An added reason for not using sequence of odd Zumkeller numbers is mainly due to point up the minimum sequence of Zumkeller numbers. Mahanta et al. [2020] stated that 945 is the smallest odd Zumkeller number. In this paper we discuss the existence of minimal Reinhard Zumkeller divisor cordial labeling of path, cycle, star $K_{1,s}$, complete bipartite, complete graph K_n for $n < 17$, tadpole graph $T_{n,k}$ for all values of n and k.

2 Preliminaries

Definition 2.1. ***Varatharajan et al. [2011]*** *Let* $G = (V,\ E)$ *be a simple graph and* $f\ :\ V\ \to\ \{1,\ 2,\ \ldots\ |V|\}$ *be a bijection. For each edge* uv*, assign the label 1 if either* $f(u) \mid f(v)$ *or* $f(v) \mid f(u)$ *and the label 0 if* $f(u) \nmid f(v)$*.* f *is called a divisor cordial labeling if* $|e_f(0) - e_f(1)| \leq 1$*. A graph with a divisor cordial labeling is called a divisor cordial graph.*

1. $e_{f^{|}}(0)$*is the number of edges of the graph G having label* 0 *under* $f^{|}$

2. $e_{f^{|}}(1)$ *is the number of edges of the graph G having label 1 under* $f^{|}$

Definition 2.2. ***Peng and Rao [2013]*** *A positive integer n is said to be a Zumkeller number if the positive divisors of n can be partitioned into two disjoint subsets of equal sum. A Zumkeller partition for a Zumkeller number n is a partition {A, B} of the set of positive divisors of n so that each of A and B sums to the same value.*

Proposition 2.3. ***Peng and Rao [2013]*** *For any prime* $p \neq 2$ *and positive integer k with* $p \leq 2^{k+1} - 1$*, the number* $2^k p$ *is a Zumkeller number.*

Fact 2.4. ***Peng and Rao [2013]*** *Let the prime factorization of an even Zumkeller number* n *be* $2^k p_1^{k_1}\, p_2^{k_2} \dots p_m^{k_m}$ *where* k *is a positive integer. Then at least one of* k_i *must be odd.*

Definition 2.5. ***Balamurugan et al. [2014]*** *A simple graph* $G = (V, E)$*, where* V *is vertex set and* E *is edge set of* G *is said to admit a Zumkeller labeling if there exists an injective function* $f : V \to N$ *such that* $f^* : E \to N$ *defined as* $f^*(xy) = f(x)f(y)$ *is a Zumkeller number for* $xy \in E; x, y \in V$*. The labelled graph G is called as a Zumkeller graph.*

Definition 2.6. ***Murali et al. [2017]*** *Let* $G = (V, E)$ *be a graph. An injective function* $f : V \to N$ *is said to be a Zumkeller cordial labeling of the graph G if there exists an induced function* $f^* : E \to \{0, 1\}$ *defined by* $f^*(xy) = f(x)f(y)$ *satisfies the following conditions*

1. *For every* $xy \in E$*,* $f^*(xy) = \begin{cases} 1\,, \ f(x)\, f(y)\, is\ a\ Zumkeller\ number\,; \\ 0\,, \ otherwise \end{cases}$

2. $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$

Definition 2.7. ***Murali et al. [2017]*** *A graph* $G = (V, E)$ *which admits a Zumkeller cordial labeling is called a Zumkeller cordial graph.*

3 Main Results

The vertex labeling γ mention in the definition 3.1 is defined by using Proposition 2.3 and fact 2.4.

Definition 3.1. *Let* $G = (V, E)$ *be a simple graph and* $\gamma : V(G) \to\ minimum \left\{2^i \times 3, 2^{j+1} \times 5, 2^{k+1} \times 7, 2^l \times 3 \times 5, 2^m \times 3 \times 7\ where\ i, j, k, l, m \geq 1\right\}$ *be an injection satisfying any one of the following conditions* $|i| + |j| = |V(G)|$ *or* $|i| + |k| = |V(G)|$ *or* $|i| + |l| = |V(G)|$ *or* $|i| + |m| = |V(G)|$ *or* $|i| + |j| + |k| =$

$|V(G)|$ or $|i|+|j|+|l| = |V(G)|$ or $|i|+|j|+|m| = |V(G)|$ or $|i|+|j|+|k|+|l| = |V(G)|$ or $|i|+|j|+|k|+|m| = |V(G)|$ or $|i|+|j|+|k|+|l|+|m| = |V(G)|$ i.e., the sum of the cardinality of exponent of $\gamma(V(G))$ should be equal to the order of the graph G. For each edge uv, assign the label 1 if $\gamma(u) \mid \gamma(v)$ or $\gamma(v) \mid \gamma(u)$ where $\gamma(u)$ and $\gamma(v)$ are Zumkeller numbers and the label 0 if $\gamma(u) \nmid \gamma(v)$ and also if $|e\gamma^{|}(0) - e\gamma^{|}(1)| \leq 1$ then γ is called minimal Reinhard Zumkeller divisor cordial labeling. A graph with a minimal Reinhard Zumkeller divisor cordial labeling is called a minimal Reinhard Zumkeller divisor cordial graph.

Theorem 3.2. *The path P_n is a minimal Reinhard Zumkeller divisor cordial when $n \equiv 0, 1\,(mod\,2)$*

Proof. Let $v_1, v_2, \ldots .v_n$ be the vertices of the path P_n. Label those consecutive adjacent vertices in the order as $2^i \times 3$ and $2^l \times 3 \times 5$ where $1 \leq i \leq \frac{n}{2}$ and $1 \leq l \leq \frac{n}{2}$ for the path having even number of vertices and for the path having odd number of vertices $1 \leq i \leq \frac{n+1}{2}$ and $1 \leq l \leq \frac{n-1}{2}$ and also $|i| + |l| = |V(G)|$. If $2^i \times 3\, |2^l \times 3 \times 5$ then the consecutive adjacent vertices contribute 1 to each edge and if $2^i \times 3 \nmid 2^l \times 3 \times 5$ then the consecutive adjacent vertices contribute 0 to each edge. Thus $e\gamma^{|}(1) = \frac{n}{2}$ and $e\gamma^{|}(0) = \frac{n-2}{2}$ if n is even and $e\gamma^{|}(1) = e\gamma^{|}(0) = \frac{n-1}{2}$ if n is odd. Hence $|e\gamma^{|}(0) - e\gamma^{|}(1)| \leq 1$.Thus P_n is a minimal Reinhard Zumkeller divisor cordial graph. □

Theorem 3.3. *The cycle C_n is a minimal Reinhard Zumkeller divisor cordial when $n \equiv 1\,(mod\,2)$, $n \geq 3$, $n \in N$*

Proof. Let $v_1, v_2, \ldots .v_n$ be the vertices of the cycle C_n. By making use of the similar pattern described as for path, the cycle of odd order is investigated as a minimal Reinhard Zumkeller divisor cordial graph. □

Theorem 3.4. *The cycle C_n admits a minimal Reinhard Zumkeller divisor cordial when the vertex v_n is labelled with $2 \times 3 \times 7$ where $n \equiv 0\,(mod\,2)$, $n \geq 4$, $n \in N$.*

Proof. Let $v_1, v_2, \ldots .v_n$ be the vertices of the cycle C_n. Label the vertex v_n with $2 \times 3 \times 7$ and appertain with the similar pattern described as for path for the remaining vertices results in a minimal Reinhard Zumkeller divisor cordial labeling for the cycle graph. □

Theorem 3.5. *The Wheel graph $W_n = K_1 + C_n$ is a minimal Reinhard Zumkeller divisor cordial*

Proof. Let v_o be the center vertex of W_n and label the center vertex as $2^2 \times 5$

Case 1. n is even.
Label the vertices $v_1, \ldots .v_n$ of C_n as $2^i \times 3$ where $1 \leq i \leq n$ such that $\gcd((20,\gamma(v_1)), (20,\gamma(v_2)), \ldots .(20,\gamma(v_n)) = 1$ and $\gcd(\gamma(v_g), \gamma(v_{g+1})) > 1$ where $1 \leq g \leq n-1$ and also $|i| + |j| = n+1 = n+1 = |V(W_n)|$.
We observe that,

$$e\gamma^{|}(0) = e\gamma^{|}(1) = n. \tag{1}$$

Case 2. n is odd.
Label the center vertex v_o as $2^2 \times 5$ and label $v_1, \ldots v_n$ of C_n with the same labeling design mentioned in case 1
Here also,

$$e\gamma^{|}(0) = e\gamma^{|}(1) = n \tag{2}$$

Hence, from (1) and (2) we get that $|e\gamma^{|}(0) - e\gamma^{|}(1)| = \begin{cases} 0\, if\ n\ is\ even \\ 0\, if\ n\ is\ odd \end{cases}$ □

Theorem 3.6. *The star graph $K_{1,s}$ is a minimal Reinhard Zumkeller divisor cordial labeling when $s \equiv 0, 1 (mod\ 2)$*

Proof. Let $G = K_{1,s}$ be the star graph with vertex set $V(G) = \{v_0 \cup \{v_g : 1 \leq g \leq s\}\}$ where $v_0$is a center vertex and v_g's are pendant vertices and an edge set $E(G) = \{e_g = v_0 v_g : 1 \leq g \leq s\}$. Here we notice that the order of the graph $|V(G)| = s + 1$ and the size of the graph $|E(G)| = s$.

Case 1. $s \equiv 1\ (mod\ 2)$
Assume $\gamma(v_0) = 2 \times 3$ which is a Zumkeller number.
The pendant vertices contribute 1 to its adjacent edges are labelled as follows. $\gamma(\mathrm{v_i}) = 2^{i+1} \times 3$ for $1 \leq i \leq \frac{s-1}{2} + 1$ and the pendant vertices contributes 0 to its adjacent edges are labelled as follows $\gamma(v_j) = 2^{j+1} \times 5$ for $1 \leq j \leq \frac{s-1}{2}$. And also $|i| + |j| = s + 1$

Case 2. $s \equiv 0\ (mod\ 2)$
Assume $\gamma(v_0) = 2 \times 3$ which is a Zumkeller number.
The pendant vertices contribute 1 to its adjacent edges are labelled as follows. $\gamma(v_i) = 2^{i+1} \times 3$ for $1 \leq i \leq \frac{s}{2}$ and the pendant vertices contributes 0 to its adjacent edges are labelled as follows $\gamma(v_j) = 2^{j+1} \times 5$ for $1 \leq i \leq \frac{s}{2}$. And also $|i| + |j| = s+1$

Hence from cases 1 and 2, we get that $e\gamma^{|}(0) = \frac{s+1}{2}$ and $e\gamma^{|}(1) = \frac{s-1}{2}$ when m is odd and $e\gamma^{|}(0) = e\gamma^{|}(1) = \frac{s}{2}$ when s is even.

Hence $\left|e\gamma^{|}(0)-\ e\gamma^{|}(1)\right| \ = \begin{cases} 0\, if\ s\, is\, even \\ 1\, if\ s\, is\, odd \end{cases}$

Thus $\left|e\gamma^{|}(0)-e\gamma^{|}(1)\right| \ \leq 1$. Hence, $k_{1,s}$ is a minimal Reinhard Zumkeller divisor cordial. □

Theorem 3.7. *The complete bipartite graph $K_{x,z}$ is a minimal Reinhard Zumkeller divisor cordial graph for all values of $x, y \geq 2$*

Proof. Let $V = V_1 \cup\ V_2$ be the bipartition of $K_{x,z}$ such that $V_1 = \{v_1, v_2, \ldots v_x\}$ and $V_2 = \{w_1, w_2, \ldots w_z\}$ .The order of the complete bipartite graph $K_{x,z}$ is $x + z \ = \ f$.

Case 1. $x = z$ where x and z are even
Obviously, there are $\frac{f}{2}$ vertices in V_1 and $\frac{f}{2}$ vertices in V_2. Then label $\frac{f}{4}$ vertices out of $\frac{f}{2}$ vertices as $2^i{\times}3$, where $1{\leq}i{\leq}\frac{f}{2}$ and the remaining $\frac{f}{4}$ vertices get the label as $2^{j+1}{\times}5$, where $1{\leq}j{\leq}\frac{f}{2}$. Label $\frac{f}{4}$ vertices in V_2 as $2^i{\times}3$, where $\frac{f}{2}{+}1{\leq}i{\leq}z$ and the remaining $\frac{f}{4}$ vertices as $2^{\mathrm{j}+1}{\times}5$ where $\frac{f}{2}{+}1{\leq}i{\leq}z$. Then the cordiality condition $|e\gamma^{|}(0) -\ e\gamma^{|}(1)| \ = \ 0$.

Case 2. $x = z$, when x and z are odd.
Label $\frac{f}{2}$ vertices in V_1 as follows: Label $\frac{\frac{f}{2}+1}{2}$ vertices out of $\frac{f}{2}$ in V_1 as $2^i{\times}3$, where $1{\leq}i{\leq}\frac{\frac{f}{2}+1}{2}$ and label the remaining vertices $\frac{f}{2} - \frac{\left(\frac{f}{2}+1\right)}{2}$ vertices are labelled as $2^{i+1}{\times}5$, where $1{\leq}j{\leq}\frac{f}{2} - \frac{\frac{f}{2}+1}{2}$. Then label $\frac{f}{2}$ vertices in V_2 as follows: Label $\frac{\frac{f}{2}+1}{2}$ vertices out of $\frac{f}{2}$ in V_2 as $2^i \times 3$, where $\frac{\left(\frac{f}{2}+1\right)}{2} + 1 \leq i \leq z+1$ and label the remaining vertices $z - \frac{\frac{f}{2}+1}{2}$ in V_2 as $2^{j+1} \times 5$, where $\frac{f}{2}\ - \frac{\left(\frac{f}{2}+1\right)}{2}\ + 1 \leq i \leq z-1$
Then the cordiality condition $|e\gamma^{|}(0) -\ e\gamma^{|}(1)| \ = \ 1$.

Case 3. $x < z$ and $x + z$ where $z = x + 1$, x is odd and z is even
There are $\frac{f+1}{2} - 1$ vertices in V_1 and $\frac{f+1}{2}$ vertices in V_2 .Label $\frac{f+1}{2} - 1$ vertices in V_1 as follows: Label $\frac{\frac{f+1}{2}}{2}$ vertices out of $\frac{f+1}{2} - 1$ in V_1 as $2^i \times 3$, where $1 \leq i \leq \frac{\frac{f+1}{2}}{2}$ and the remaining vertices $\frac{f+1}{2} - 1\ - \frac{\frac{f+1}{2}}{2}$ are labelled as $2^{j+1} \times 5$, where $1 \leq j \leq \frac{f+1}{2} - 1\ - \frac{\frac{f+1}{2}}{2}$ .Label $\frac{f+1}{2}$ vertices in V_2 as follows: Label $\frac{\frac{f+1}{2}}{2}$ vertices as $2^i \times 3$, where $\frac{\frac{f+1}{2}}{2} + 1 \leq i \leq z$ and label the left over vertices $\frac{f+1}{2}\ - \frac{\frac{f+1}{2}}{2}$ as $2^{j+1} \times 5$, where $\frac{f+1}{2}\ - \frac{\frac{f+1}{2}}{2} \leq j \leq z-1$. Following the labeling pattern results in $|e\gamma^{|}(0) -\ e\gamma^{|}(1)| \ = 1$

Case 4. x<z and $z = x + 1$, where x is even and z is odd.
There are $\frac{f+1}{2}{-}1$ vertices in V_1 and $\frac{f+1}{2}$ vertices in V_2. Label $\frac{f+1}{2} - 1$ vertices in V_1 as follows: Label $\frac{\frac{f+1}{2}-1}{2}$ vertices as $2^i \times 3$, where $1 \leq i \leq \frac{\frac{f+1}{2}-1}{2}$ and label

$\frac{f+1}{2} - 1 - \frac{\frac{f+1}{2}-1}{2}$ vertices as $2^{j+1} \times 5$, where $1 \le j \le \frac{\frac{f+1}{2}-1}{2}$ .Now proceed to label $\frac{f+1}{2}$ vertices in V_2 as follows: Label $\frac{\frac{f+1}{2}-1}{2} + 1$ vertices as $2^i \times 3$, where $\frac{\frac{f+1}{2}-1}{2} + 1 \le i \le z$ and label the left over vertices $\frac{f+1}{2} - \frac{\frac{f+1}{2}-1}{2} + 1$ as $2^{j+1} \times 5$, where $\frac{f+1}{2} - \frac{\frac{f+1}{2}-1}{2} + 1 \le j \le z-1$. Then the cordiality condition $\left|e\gamma^{|}(0) - e\gamma^{|}(1)\right| = 0$.

Case 5. $x > z$ and $x = z + 2$.
Obviously, there are $\frac{f}{2}+1$ vertices in V$_1$ and $\frac{f}{2} - 1$ vertices in V$_2$. Then label $\frac{\frac{f}{2}+1}{2}$ vertices out of $\frac{f}{2} + 1$ vertices as $2^i \times 3$, where $1 \le i \le \frac{\frac{f}{2}+1}{2}$ and the remaining $\frac{\frac{f}{2}+1}{2}$ vertices get the label as $2^{j+1} \times 5$, where $1 \le j \le \frac{\frac{f}{2}+1}{2}$. Likewise label $\frac{f}{2} - 2$ vertices out of $\frac{f}{2} - 1$ in V$_2$ as $2^i \times 3$, where $\frac{\frac{f}{2}+1}{2} + 1 \le i \le z + 1$ and the remaining $\frac{f}{2} - 1 - \frac{f}{2} - 2$ vertices as $2^{j+1} \times 5$ where $\frac{\frac{f}{2}+1}{2} + 1 \le j \le z + 1$. Then the cordiality condition $\left|e\gamma^{|}(0) - e\gamma^{|}(1)\right| = 0$.

Proceeding like this for all values of x and z, the cordiality condition is satisfied. Hence the complete bipartite is a minimal Reinhard Zumkeller divisor cordial graph. □

Theorem 3.8. *The tadpole $T_{n,\,k}$ is a minimal Reinhard Zumkeller divisor cordial graph for all values of n and k*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C$_n$ and $w_1, w_2, \ldots, w_k$ be the vertices of the path P_k. Let $T_{n,k}$ be the repercussion graph obtained by recognizing a vertex of cycle C_n to an end vertex of the path P_k .Then the order of $T_{n,k}$ graph is $|V(T_{n,k})| = n+k$ and the size of $T_{n,k}$ graph is $|E(T_{n,k})| = n+k$. Concatenate the pendant vertex of P_k to one of the vertices of C_n with an edge in such a way that $\begin{cases} v_{\frac{n+3}{2}=w_1} \; for\ n \equiv 1\,(mod\ 2)\ if\ \frac{n+3}{2}\ is\ even \\ v_{\frac{n+3}{2}-1=w_1} \; for\ n \equiv 1\,(mod\ 2)\ if\ \frac{n+3}{2}\ is\ odd \\ \frac{v_{n+2}}{2} = w_1\ for\ n \equiv 0\,(mod\ 2)\,if\ \frac{n+2}{2}\ is\ even \\ \frac{v_{n+2}}{2} - 1 = w_1\ for\ n \equiv 0\,(mod\ 2)\ if\frac{n+2}{2}\ is\ odd \end{cases}$.
We contemplate the following cases.Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle $C_n$be labelled as follows:

Case 1. $n \equiv 1\,(mod\ 2)$ and $K = 1$.
Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle $C_n$be labelled as follows:

$$\gamma(v_g) = \begin{cases} 2^i \times 3\ where\ g \equiv 1\,(mod\ 2)\ and\ 1 \le i \le \frac{n+1}{2} \\ 2^l \times 3 \times 5\ where\ g \equiv 0\,(mod\ 2)\ and\ 1 \le l \le \frac{n+1}{2} - 1 \end{cases} \tag{3}$$

Let the vertices of the path P_h w_1, w_2 be labelled as follows.

$$\gamma\left(v_{\frac{n+3}{2}}\right) = \gamma(w_1) \tag{4}$$

$$\gamma(w_2) = 2^{\frac{n+1}{2}+1} \times 3 \tag{5}$$

In regards to the labeling designs (3),(4),(5), we get that $e^{|}_{\gamma}(0) \;= \frac{n+K}{2}$; $e^{|}_{\gamma}(1) = \frac{n+K}{2}$. Hence $\left|e^{|}_{\gamma}(0) - \; e^{|}_{\gamma}(1)\right| \; = 0$.

Case 2. $n \equiv 1\,(mod\,2)$ and $k \geq 2$, where k is even

$$\gamma(v_g) = \begin{cases} 2^i \times 3 \; where \; g \equiv 1\,(mod\,2)\, and\; 1 \leq i \leq \frac{n+1}{2} \\ 2^l \times 3 \times 5 \; where \; g \equiv 0\,(mod\,2) \;\, and \; 1 \leq l \leq \frac{n+1}{2} - 1 \end{cases} \tag{6}$$

$$\gamma(w_h) = \begin{cases} 2^i \times 3 \; where \; h \equiv 1\,(mod\,2)\,, \; 3 \leq h \leq k+1 \; and \frac{n+1}{2} + 1 \leq \; i \leq \frac{n+1+k}{2} \\ 2^l \times 3 \times 5 \; where \; h \equiv 0\,(mod\,2)\,, \; 2 \leq h \leq k \; and \frac{n+1}{2} \leq \; l \leq \frac{n+1+k}{2} - 1 \end{cases} \tag{7}$$

Hence from (6) and (7), we get that $e^{|}_{\gamma}(1) = \frac{n+k+1}{2}$; $e^{|}_{\gamma}(0) = \; \frac{n+k-1}{2}$.
Hence $\left|e^{|}_{\gamma}(0) - \; e^{|}_{\gamma}(1)\right| \; = 1$.

Case 3. $n \equiv 0\,(mod\,2)$ and $k \geq 1$ where k is odd

$$\gamma(v_g) = \begin{cases} 2^i \times 3 \; where \; g \equiv 1\,(mod\,2)\, and\; 1 \leq i \leq \frac{n}{2} \\ 2^l \times 3 \times 5 \; where \; g \equiv 0\,(mod\,2) \;\, and\; 1 \leq l \leq \frac{n}{2} \end{cases} \tag{8}$$

Let the vertices of the path P_h, $w_1, w_2, \ldots, w_h$ be labelled as follows,

$$\gamma(w_h) = \begin{cases} 2^i \times 3 \; where \; h \equiv 0\,(mod\,2)\,, \; 2 \leq h \leq k+1 \; and \; \frac{n}{2} + 1 \leq \; i \leq \frac{n+k+1}{2} \\ 2^l \times 3 \times 5 \; where \; h \equiv 1\,(mod\,2)\,, \; 3 \leq h \leq k \; and \; \frac{n}{2} + 1 \leq \; l \leq \frac{n+k+1}{2} - 1 \end{cases} \tag{9}$$

In regards to the above labeling design (8) and (9), we get that $e^{|}_{\gamma}(1) = \frac{n+k-1}{2}$; $e^{|}_{\gamma}(0) = \frac{n+k+1}{2}$
Hence $\left|e^{|}_{\gamma}(0) - \; e^{|}_{\gamma}(1)\right| \; = 1$.

Case 4. $n \equiv 1\,(mod\,2)$ where $k \; \geq 1$ where k is odd

$$\gamma(v_g) = \begin{cases} 2^i \times 3 \; where \; g \equiv 1\,(mod\,2)\, and\; 1 \leq i \leq \frac{n+1}{2} \\ 2^l \times 3 \times 5 \; where \; g \equiv 0\,(mod\,2) \;\, and\; 1 \leq l \leq \frac{n+1}{2} - 1 \end{cases} \tag{10}$$

$$\gamma(w_h) = \begin{cases} 2^i \times 3 \; where \; h \equiv 1\,(mod\,2)\,, \; 3 \leq h \leq k \; and \; \frac{n+1}{2} + 1 \leq \; i \leq \frac{n+k}{2} \\ 2^l \times 3 \times 5 \; where \; h \equiv 0\,(mod\,2)\,, \; 2 \leq h \leq k-1 \; and \; \frac{n+1}{2} + 1 \leq \; l \leq \frac{n+k}{2} - 1 \end{cases} \tag{11}$$

$$\gamma(w_{k+1}) = 2^2 \times 5 \tag{12}$$

In regards to the labeling design (10), (11) and (12), we get that $e^{|}_{\gamma}\,(1) = \frac{n+k}{2}$;
$e^{|}_{\gamma}\,(0) = \frac{n+k}{2}$
Hence $\left|e^{|}_{\gamma}\,(0) - \; e^{|}_{\gamma}\,(1)\right| = 0$

Case 5. $n \equiv 0\,(mod\,2)$ where $K\ \geq 2$ and K is even

$$\gamma(v_g) = \begin{cases} 2^i \times 3\ where\ g \equiv 1\,(mod\,2)\ and\ 1 \leq i \leq \frac{n}{2} \\ 2^l \times 3 \times 5\ where\ g \equiv 0\,(mod\,2)\ \ and\ 1 \leq l \leq \frac{n}{2} \end{cases} \tag{13}$$

$$\gamma(w_h) = \begin{cases} 2^i \times 3\ where\ h \equiv 0\,(mod\,2)\,,\ 2 \leq h \leq k\ and\ \frac{n}{2}+1 \leq\ i \leq \frac{n+k}{2} \\ 2^l \times 3 \times 5\ where\ h \equiv 1\,(mod\,2)\,,\ 3 \leq h \leq k-1\ and\ \frac{n}{2}+1 \leq\ l \leq \frac{n+k}{2}-1 \end{cases} \tag{14}$$

$$\gamma(w_{k+1}) = 2^2 \times 5 \tag{15}$$

In regards to the labeling design (13), (14) and (15), we get that $e_\gamma^|\,(1) = \frac{n+k}{2}$; $e_\gamma^|\,(0) = \frac{n+k}{2}$ Hence $\left|e_\gamma^|\,(0) -\ e_\gamma^|\,(1)\right| = 0$

Hence from all cases we get that the tadpole $T_{n,k}$ is a minimal Reinhard Zumkeller divisor cordial graph for all values of n and k. $\square$

Theorem 3.9. *The complete graph K_n is a minimal Reinhard Zumkeller divisor cordial if $n \leq 16$*

Proof. Obviously $K_1,\ K_2,\ K_3$ are minimal Reinhard Zumkeller divisor cordial graph. The following table 1 brings forth a minimal Reinhard Zumkeller divisor cordial labeling of K_n for $4 \leq n < 17$

Order of K_n	Vertex labels	Cordiality Condition
4	6,12,20,24	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 0$
5	6,12,28,30,60	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 0$
6	6,12,24,28,30,60	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 1$
7	6,12,24,28,30,48,60	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 1$
8	6,12,24,28,30,48,60,96	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 0$
9	6,12,24,28,30,48,60,96,168	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 0$
10	6,12,24,28,30,48,60,96,168,192	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 1$
11	6,12,24,28,30,48,60,84,96,168,192	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 1$
12	6,12,24,28,30,48,60,84,96,120,168,192	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 0$
13	6,12,24,28,30,48,60,84,96,120,168,192,384	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 1$
14	6,12,24,28,30,48,60,84,96,120,168,192,240,384	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 1$
15	6,12,24,28,30,48,60,84,96,120,168,192,240,384,768	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 1$
16	6,12,24,28,30,48,60,84,96,120,168,192,240,336,384,768	$\left\|e\gamma^|\,(0) -\ e\gamma^|\,(1)\right\| = 0$

Table 1: Minimal Reinhard Zumkeller divisor cordial labeling of $K_n$for $4 \leq n < 17$

K_{17} is not Reinhard Zumkeller divisor cordial graph. Since by following the labeling pattern of K_{16} the vertex labels to be selected for the vertex v_{17} must be anyone of them: 480 or 1536 or 672 or by choosing 20 instead of 28 from the above labeling pattern then the cordiality condition is $\left|e\gamma^{|}(1) - e\gamma^{|}(0)\right| = 69 - 67 = 2$ and $\left|e\gamma^{|}(0) - e\gamma^{|}(1)\right| = 69 - 67 = 2$ respectively. Since the labeling pattern for each complete graph K_n follows the labeling pattern of its predecessor, for all higher order complete graphs the cordiality condition increases by 1 for each $n \geq 17$. □

4 Discussion

For the notion of minimal Reinhard Zumkeller divisor cordial labeling ,this effort has produced several fresh findings. In order to create a minimal Reinhard Zumkeller divisor cordial graph and introduce a new element to the labeling pattern of various graph structures, the traits of the Zumkeller number are unified with the divisor cordial graph labeling technique. The results that are established in this paper are amalgamated and motivated us to get into the conclusion that for every connected minimal Reinhard Zumkeller divisor cordial graph G, $\gamma(u) \equiv 0(mod6)$ for some vertex $u \in V(G)$. Deriving similar results for other graph families is an open problem .

5 Conclusions

In the present investigation, minimal Reinhard Zumkeller divisor cordial labeling has been introduced and probed for the existence of Reinhard Zumkeller divisor cordial labeling of path, cycle, star $K_{1,s}$, complete graph K_n for $n < 17$, complete bipartite and tadpole graph $T_{n,k}$ for all values of n and k. In future research work, we will develop findings to construct dense minimal Reinhard Zumkeller divisor cordial graphs, book graphs with polygonal pages, generalized Petersen graphs, wheel graphs and product related graphs.

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