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Polygonal Graceful Labeling of Some Simple Graphs

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Abstract

Let $G = (V, E)$ be a graph with p vertices and q edges. Let $V(G)$ and $E(G)$ be the vertex set and edge set of G respectively. A polygonal graceful labeling of a graph G is an injective function $\eta: V(G) \to Z^+$, where Z^+ is a set of all non-negative integers that induces a bijection $\eta^*: E(G) \to \{P_s(1), P_s(2), \ldots, P_s(q)\}$, where $P_s(q)$ is the q^{th} polygonal number such that $\eta^*(uv) = |\eta(u) - \eta(v)|$ for every edge $e = uv \in E(G)$. A graph which admits such labeling is called a polygonal graceful graph. For $s = 3$, the above labeling gives triangular graceful labeling. For $s = 4$, the above labeling gives tetragonal graceful labeling and so on. In this paper, polygonal graceful labeling is introduced and polygonal graceful labeling of some simple graphs is studied.

Keywords: Polygonal numbers, Polygonal graceful labeling, Polygonal graceful graph.

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1. Introduction

We shall consider a simple, undirected and finite graph $G = (V, E)$ on $p = |V|$vertices and $q = |E|$edges. For all standard terminology, notations and basic definitions, we follow Harary [2] and for number theory, we follow Apostal [1]. A graph labeling is an assignment of integers to the vertices or the edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (edges/both) then the labeling is called a vertex (edge/ total) labeling. Rosa [6] introduced β–valuation of a graph. Golomb [4] called it as graceful labeling. For a detailed survey of graph labeling, one can refer Gallian [3]. Ramesh and Syed Ali Nisaya [5] introduced some more polygonal graceful labeling of path. Here, we shall recall some definitions which are used in this paper.

2. Preliminaries

Definition 2.1. The **star** graph $K_{1,n}$of order $n + 1$ is a tree on n edges with one vertex having degree n and other vertices having degree 1.

Definition 2.2. A graph $S(G)$ which can be obtained from a given graph by breaking up each edge into one (or) more segment by inserting intermediate vertices between its two ends is called **subdivision graph**. It is denoted by $S(G)$.

Definition 2.3. A **path** $P_n$is obtained by joining$u_i$to the consecutive verticesu_{i+1} for $1 \leq i \leq n - 1$.

Definition 2.4. A **coconut tree** $CT(n, m)$ is a graph obtained from the path P_n by appending m new pendant edges at an end vertex.

Definition 2.5. Let P_2 be a path on two vertices and let u and v be the vertices of P_2. From u, there are m pendant vertices say $u_1, u_2, \ldots, u_m$and from v, there are n pendant vertices say$v_1, v_2, \ldots, v_n$. The resulting graph is a **bistar**$B_{m,n}$.

Definition 2.6. A **graceful labeling** of a graph G is an injective function $f: V(G) \to \{0,1,2, \ldots, q\}$ that induces a bijection $f^*: E(G) \to \{1,2, \ldots, q\}$ of the edges of G defined by $f^*(e) = |f(u) - f(v)|, \forall e = uv \in E(G)$. The graph which admits such a labeling is called a **graceful graph**.

Definition 2.7. A **polygonal number** is a number represented as dots (or) pebbles arranged in the shape of regular polygon. If s is the number of sides in a polygon, the formula for the $n^{th}s$–gonal number $P_s(n)$ is $P_s(n) = \frac{(s-2)(n)^2-(s-4)(n)}{2}$. For $s = 3$ gives triangular numbers. For $s = 4$ gives tetragonal numbers and so on.

3. Main results

Definition 3.1: Let $G = (V, E)$ be a graph with p vertices and q edges. Let $V(G)$ and$E(G)$ be the vertex set and edge set of Grespectively. A polygonal graceful labeling of a graph G is an injective function $\eta: V(G) \to Z^+$, where Z^+ is a set of all non-negative integers that induces a bijection $\eta^*: E(G) \to \{P_s(1), P_s(2), \ldots, P_s(q)\}$, where $P_s(q)$ is the q^{th} polygonal number such that $\eta^*(uv) = |\eta(u) - \eta(v)|$ for every edge $e = uv \in E(G)$. A graph which admits such labeling is called a polygonal graceful graph. For $s = 3$, the above labeling gives triangular graceful labeling. For $s = 4$, the above labeling gives tetragonal graceful labeling and so on.

Example 3.2: The hexagonal graceful labeling of $K_{1,5}$is given in figure (1)

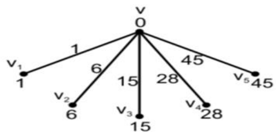

Figure (1)

Theorem 3.3: The star graph $K_{1,n}$ admits polygonal graceful labeling.
Proof: Consider the star graph $K_{1,n}$. Let $V(k_{1,n}) = \{v, v_i: 1 \leq i \leq n\}$ and$E(k_{1,n}) = \{vv_i: 1 \leq i \leq n\}$.
Then $K_{1,n}$ has $n+1$ vertices and n edges. Define $\eta: V(K_{1,n}) \to \{0,1,2, \ldots, P_s(n)\}$ as follows.

$$\eta(v) = 0$$

$$\eta(v_i) = \frac{(s-2)i^2 - (s-4)i}{2}, \quad 1 \leq i \leq n$$

$= P_s(i), 1 \leq i \leq n$
Clearly η is injective and the induced edge labels are the first n polygonal numbers. Hence the star graph $K_{1,n}$ admits polygonal graceful graph.

Example 3.4: The pentagonal graceful labeling of $K_{1,7}$ is shown in the following figure (2)

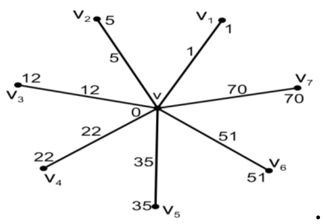

Figure (2)

Theorem 3.5: $S(K_{1,n})$, the subdivision of the star $K_{1,n}$ admits polygonal graceful labeling.
Proof: Consider the subdivision of the star graph $S(K_{1,n})$. Let $V(S(K_{1,n})) = \{v, v_i, u_i: 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{vv_i, v_iu_i: 1 \leq i \leq n\}$. Thus $S(K_{1,n})$ has $2n + 1$ vertices and $2n$ edges. Define $\eta: V(S(K_{1,n})) \to \{0,1,2, \ldots, P_s(2n)\}$ as follows
$\eta(v) = 0$

$$\eta(v_i) = \frac{(s-2)(2n-i+1)^2 - (s-4)(2n-i+1)}{2}, 1 \le i \le n$$
$$= P_s(2n-i+1), 1 \le i \le n$$
$$\eta(u_i) = \eta(v_i) + \frac{(s-2)(n-i+1)^2 - (s-4)(n-i+1)}{2}, 1 \le i \le n$$
$$= \eta(v_i) + P_s(n-i+1), 1 \le i \le n$$

Clearly η is injective and the induced edge labels are the first $2n$ polygonal numbers. Hence $S(K_{1,n})$ admits polygonal graceful graph.

Example 3.6: The triangular graceful labeling of $S(K_{1,3})$ is shown in the following figure (3).

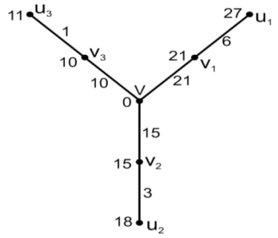

Figure (3)

Theorem 3.7: The path of length n is a polygonal graceful graph for all $n \ge 3$.

Proof. Let $G = (V, E)$ be the path of length n with the vertex set $V = \{v_1, v_2, \dots, v_n\}$and the edge set $E = \{v_i v_{i+1}/1 \le i \le n-1\}$. Then G has n vertices and $n-1$ edges. Define $\eta: V(G) \to \{0,1,2,\dots,P_s(n-1)\}$ as follows.

$$\eta(v_{2i-1}) = (i-1)((s-2)n - (s-2)i + 1), i = 1,2,.., \left\lfloor \frac{n+1}{2} \right\rfloor$$
$$\eta(v_{2i}) = \frac{1}{2}(n-1)[(s-2)n - 2s + 6] - [(s-2)n - (s-2)i - (s-3)](i-1), i = 1,2,.., \left\lfloor \frac{n}{2} \right\rfloor$$

Clearly η is injective and the induced edge labels are the first $n-1$ polygonal numbers. Hence the path of length n is a polygonal graceful graph for all $n \ge 3$.

Example 3.8: The heptagonal graceful labeling of path of length 6 is given in figure (4).

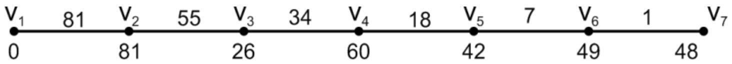

Figure (4)

Theorem 3.9: Coconut tree $CT(n,m)$ is a polygonal graceful labeling$\forall n \ge 1, m \ge 2$.

Proof: Consider the graph coconut tree $CT(n,m)$. Let $V(CT(n,m)) = \{v, v_i, u_j : 1 \le i \le n, 1 \le j \le m-1\}$ and $E(CT(n,m)) = \{vv_i, vu_1, u_j u_{j+1} : 1 \le i \le n, 1 \le j \le m-1\}$. Then $CT(n,m)$ has $m+n$ vertices and $m+n-1$ edges. Define$\eta: V(CT(n,m)) \to \{0,1,2,\dots,P_s(m+n-1)\}$ as follows.

$$\eta(v) = 0$$
$$\eta(v_i) = \frac{(s-2)(m+n-i)^2 - (s-4)(m+n-i)}{2}, 1 \le i \le n$$

$= P_s(m+n-i), 1 \le i \le n$

$$\eta(u_1) = \frac{(s-2)(m-1)^2-(s-4)(m-1)}{2}$$

$= P_s(m-1)$

$$\eta(u_j) = \begin{cases} \eta(u_{j-1}) + \frac{(s-2)(m-j)^2-(s-4)(m-j)}{2}, if j is odd and 2 \le j \le m-1 \\ \eta(u_{j-1}) - \frac{(s-2)(m-j)^2-(s-4)(m-j)}{2}, if j is even and 2 \le j \le m-1 \end{cases}$$

$$\eta(u_j) = \begin{cases} \eta(u_{j-1}) + P_s(m-j), \text{ if } j \text{ is odd and } 2 \le j \le m-1 \\ \eta(u_{j-1}) - P_s(m-j), \text{ if } j \text{ is even and } 2 \le j \le m-1 \end{cases}$$

Clearly η is an injective function. Let η^* be the induced edge labeling of η. Then

$$\eta^*(vv_i) = \frac{(s-2)(n+m-i)^2-(s-4)(n+m-i)}{2}, 1 \le i \le n$$

$= P_s(n+m-i), 1 \le i \le n$

$$\eta^*(vu_1) = \frac{(s-2)(m-1)^2-(s-4)(m-1)}{2}$$

$= P_s(m-1)$

$$\eta^*(u_ju_{j+1}) = \frac{(s-2)(m-j-1)^2-(s-4)(m-j-1)}{2}, 1 \le j \le m-2$$

$= P_s(m-j-1), 1 \le j \le m-2$

Hence the induced edge labels are the first $m+n-1$polygonal numbers. Hence $CT(n,m)$ is a polygonal graceful graph.

Example 3.10. The octagonal graceful labeling of coconut tree $CT(5,6)$ is given in the following figure (5).

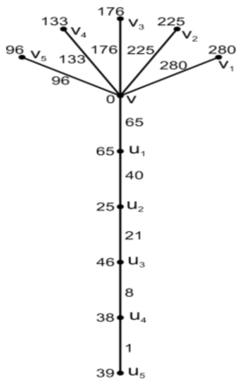

Figure (5)

Theorem 3.11. The bistar $B_{m,n}$ admits polygonal graceful labeling.

Proof.: Consider the graph $B_{m,n}$. Let $V(B_{m,n}) = \{u, v, u_i, v_j / 1 \le i \le m, 1 \le j \le n\}$ and $E(B_{m,n}) = \{uv, uu_i, vv_j / 1 \le i \le m, 1 \le j \le n\}$. Then $B_{m,n}$ has $m+n+2$ vertices and $m+n+1$ edges. Define$\eta: V(B_{m,n}) \rightarrow \{0,1,2,\dots,P_s(m+n+1)\}$ by $\eta(u) = 0$

$$\eta(v) = \frac{(s-2)(m+n+1)^2 - (s-4)(m+n+1)}{2}$$

$$= P_s(m+n+1)\eta(u_i) = \frac{(s-2)(m+n-i+1)^2 - (s-4)(m+n-i+1)}{2}, 1 \leq i \leq m$$

$$= P_s(m+n-i+1), 1 \leq i \leq m$$

$$\eta(v_j) = \eta(v) - \frac{(s-2)(n-j+2)^2 - (s-4)(n-j+2)}{2}, 1 \leq j \leq n$$

$$= \eta(v) - P_s(n-j+2), 1 \leq j \leq n$$

Clearly η is injective and the induced edge labels are the first $m+n+1$polygonal numbers. Hence $B_{m,n}$admits polygonal graceful graph.

Example 3.12: The hexagonal graceful labeling of $B_{5,4}$ is given in the following figure (6).

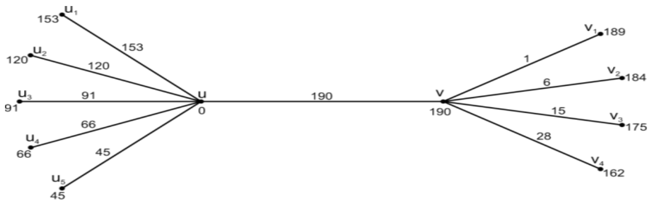

Figure (6)

4. Conclusion

In this paper, polygonal graceful labeling is introduced and polygonal graceful labeling of some simple graphs is studied. This work contributes several new results to the theory of graph labeling**.**

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