The Forcing Geodetic Cototal Domination Number of a Graph

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Abstract

Let S be a geodetic cototal domination set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum geodetic cototal domination set containing T. The minimum cardinality T is the forcing geodetic cototal domination number of S is denotedby $f_{\gamma gct}(S)$, is the cardinality of a minimum forcing subset of S. The forcing geodetic cototal domination number of G,denoted by $f_{\gamma gct}(S)$, is $f_{\gamma gct}(G) = min\{f_{\gamma gct}(S)\}$, where the minimum is takenover all γ_{gct}-sets S in G. Some general properties satisfied by this concept arestudied. It is shown that for every pair a, b of integers with $0 \leq a < b, b \geq 2$,there exists a connected graph G such that $f_{\gamma gct}(G) = a$ and $\gamma_{gct}(G) = b$. where$\gamma_{gct}(G)$ isthe geodetic cototal dominating number of G.

Keywords: geodetic set, cototal dominating set, geodetic cototal dominating set, geodetic cototal domination number, forcing geodetic cototal domination number.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by $m$and $n$respectively. For basic definitions and terminologies, we refer to [1,2]. For vertices u and v in a connected graph G, the distance$d(u, v)$ is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u, v)$ is called a *$u-v$geodesic*. The *eccentricity$e(v)$* of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, $radG$ or$r(G)$ and the maximum eccentricity is its *diameter*, $diamG$of G. Let $x, y \in V$and let$I[x, y]$ be the set of all vertices that lies in $x - y$ geodesic including $x$and y. Let $S \subseteq V(G)$and $I[S] = \cup_{x,y \in S} I[x, y]$. Then S is said to be a *geodetic set* of G, if $I[S] = V$. The *geodetic number$g(G)$* of $G$is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is called a *g-set*of G. A set $S \subseteq V(G)$ is called a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex of S. The *domination number*, $\gamma(G)$, of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as a γ-set of G. The domination concept was studied in [3]. A dominating set S of G is a *cototal dominating set* if every vertex $v \in V \setminus S$ is not an isolated vertex in the induced subgraph $\langle V \setminus S \rangle$. The cototal domination number $\gamma_{ct}(G)$ of G is the minimum cardinality of a cototal dominating set. The cototal domination number of a graph was studied in [4]. A set $S \subseteq V$ is said to be a geodetic cototal dominating set of G, IfS is both geodetic set and cototal dominating set of G. The geodetic cototal domination number of G is the minimum cardinality among all geodetic cototal dominatingsets in G and denoted by $\gamma_{gct}(G)$. A geodetic cototal dominating set of minimumcardinality is called the γ_{gct}-set of G. The geodetic cototal domination number of agraph was studied in [6]. The following theorems are used in the sequel.

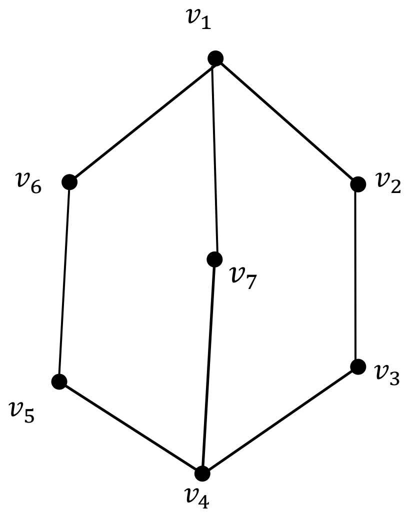
Theorem 1.1. [6] Every end vertex of G belongs to every geodetic cototal dominating set of G.

2. The forcing geodetic Cototal domination number of a graph

Even though every connected graph contains a minimum geodetic cototal dominating sets, some connected graph may contain several minimum geodetic cototal dominating sets. For each minimum geodetic cototal dominating set S in a connected graph there is always some subset T of S that uniquely determines S as the minimum geodetic cototal dominating set containing T such “forcing subsets” are considered in this section. The forcing concept was studied in [5]

Definition 2.1. Let S be a geodetic cototal domination set of G. A subset T ⊆ S is called a forcing subset for S if S is the unique minimum geodetic cototal domination set containing T. The minimum cardinality T is the forcing geodetic cototal domination number of S is denoted by $f_{\gamma gct}(S)$, is the cardinality of a minimum forcing subset of S. The forcing geodetic cototal domination number of G, denoted by$f_{\gamma gct}(S)$, is $f_{\gamma gct}(G)$ = min $\{f_{\gamma gct}(S)\}$, where the minimum is taken over all γ_{gct}-setsS in G.

Example 2.2. For the graph G of Figure 2.1, $S_1 = \{v_3, v_6, v_7\}$ and $S_2 = \{v_2, v_5, v_7\}$are the only two γ_{gct}-sets of G so that $\gamma_{gct}(G) = 3$ and $f_{\gamma gct}(S_1) = f_{\gamma gct}(S_2) = 1$ sothat $f_{\gamma gct}(G) = 1$.



G
Figure 2.1

The following result follows immediately from the definitions of the geodetic cototal domination number and the forcing geodetic cototal domination number of a connected graph G.

Theorem 2.3. For every connected graph G, $0 \leq f_{\gamma gct}(G) \leq \gamma_{gct}(G)$.

Remark 2.4. The bounds in Theorem 2.3 are sharp. For the complete graph $G = K_n$,$S = V$ is the unique γ_{gct}-set of G so that $f_{\gamma gct}(G) = 0$. Also, the bounds in Theorem2.3 can be strict. For the graph G given in Figure 2.1, $\gamma_{gct}(G)$= 3 and $f_{\gamma gct}(G)$ = 1. Thus $0 < f_{\gamma gct}(G) < \gamma_{gct}(G)$.

Theorem 2.5. Let G be a connected graph. Then
(a) $f_{\gamma gct}(G) = 0$ if and only if G has a unique minimum γ_{gct}-set.
(b) $f_{\gamma gct}(G) = 1$ if and only if G has at least two minimum γ_{gct}-sets, one of which isa unique minimum γ_{gct}-set containing one of its elements and
(c) $f_{\gamma gct}(G) = \gamma_{gct}(G)$ if and only if no γ_{gct}-set of G is the unique minimum γ_{gct}-set containing any of its proper subsets.

Definition 2.6. A vertex v of a connected graph G is said to be a geodetic cototal dominating vertex of G if v belongs to every γ_{gct}-set of G.

Example 2.7. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_3, v_6\}$ and $S_2 = \{v_1, v_3, v_5\}$ are the only two minimum γ_{gct}-sets of G so that $\{v_1, v_3\}$ is the geodeticco-total dominating vertex of G.Then $f_{\gamma gct}(G) \leq \gamma_{gct}(G) - |W|$.

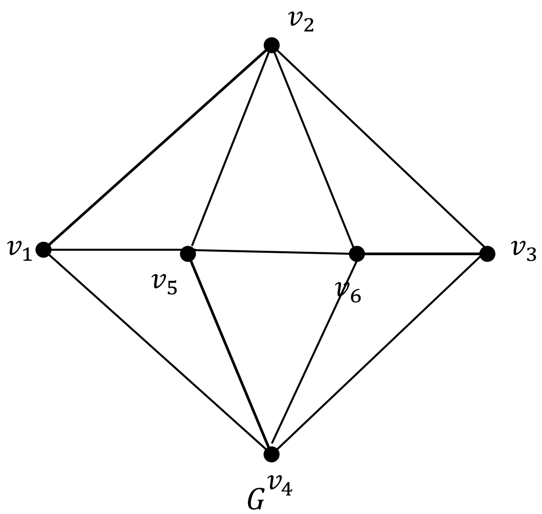


Figure 2.2

Remark 2.9. The bound in Corollary 2.7 is sharp. For the graph G of Figure 2.2, $S_1 = \{v_1, v_3, v_6\}$ and $S_2 = \{v_1, v_3, v_5\}$ are the only two minimum γ_{gct}-sets of G so that $f_{\gamma gct}(S_1) = f_{\gamma gct}(S_2) = 1$ so that $\gamma_{gct}(G) = 3$ and $f_{\gamma gct}(G) = 1$. Also, $W = \{v_1, v_3\}$ is theset of all geodetic cototal dominating vertices of G. Now,$\gamma_{gct}(G) - |W| = 3 - 2 = 1$. Thus $f_{\gamma gct}(G) < \gamma_{gct}(G) - |W|$. Also, the bounds in Theorem 2.7 can be strict.

Theorem 2.10. For the complete bipartite graph $G = K_{r,s}$ $(1 \leq r \leq s)$,

$$f_{\gamma gct}(G) = \begin{cases} 0, & if\ 1 \leq r \leq 3 \\ 4, & if 4 \leq r \leq s \end{cases}$$

Proof: Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_s\}$ be the bipartite sets of G. For $1 \leq r \leq 3$. Let $S = U \cup W$ is the unique γ_{gct}-set of G so that $f_{\gamma gct}(G) = 0$. Let$1 \leq r \leq 3$. If $r \geq 4$, then every $\gamma_{gct}(G)$-set is of the form $S = \{u_{i_1}, u_{i_2}, w_{j_1}, w_{j_2}\}$where$1 \leq i_1 \leq i_2 \leq r$ and $1 \leq j_1 \leq j_2 \leq s$. Since S is not the unique geodetic cototaldominating set containing any of its proper subset, By Theorem $f_{\gamma gct}(G) = 4$.■

Theorem 2.11. For the wheel $G = K_n + C_{n-1}$ $(n \geq 5)$,

$$f_{\gamma gct}(G) = \begin{cases} 1, & ifniseven \\ 2, & ifnisodd \end{cases}$$

Proof: Let x be the central vertex of G and C_{n-1} be $v_1, v2, \ldots, v_{n-1}, v_n$.
Case 1: n is even.
Then $S_1 = \{v_1, v_3, v_5, \ldots, v_{n-3}, v_{n-1}\}, S_2 = \{v_2, v_4, v_6, \ldots, v_{n-2}, v_n\}$ are the only twoγ_{gct}-sets of G such that $f_{\gamma gct}(S_1) = f_{\gamma gct}(S_2) = 1$ so that $f_{\gamma gct}(G) = 1$.
Case 2: n is odd.
Then $S_1 = \{v_1, v_3, v_5, \ldots, v_n\}, S_2 = \{v_2, v_4, v_6, \ldots, v_{n-1}, v_1\}, \ldots, S_{n/2} = \{v_{n/2}, v_{n/2+1}, \ldots, v_1, v_3, v_{n/2-1}\}$are the $n/2$ γ_{gct}-sets of G such that $f_{\gamma gct}(S_1) = f_{\gamma gct}(S_2) = \ldots = 4 f_{\gamma gct}(S_{n/2}) = 2$ sothat $f_{\gamma gct}(G) = 2$.■

Theorem 2.12. For the helm graph $G = H_r, G = T, f_{\gamma gct}(G) = 0$, for $n \geq 6$.
Proof: Let S be the set of end vertices and the cut vertices of G. Then S is theunique γ_{gct}-set of G so that $f_{\gamma gct}(G) = 0$.■

Theorem 2.13. For the Triangular snake graph $G = T_r$, $f_{\gamma gct}(G) = 0$.
Proof: Let S be the set of extreme vertices of G. Then S is the unique γ_{gct}-set of Gso that $f_{\gamma gct}(G) = 0$.■

Theorem 2.14. For the fan graph $F_n = K_1 + P_{n-1}$,

$$f_{\gamma gct}(G) = \begin{cases} 0, & if n-1\ isodd \\ 1, & ifniseven \end{cases}$$

Proof: Let $V(K_1) = \{x\}$ and $V(P_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$.
Let $n - 1$ is odd. Let $n - 1 = 2k + 1$. Then $S = \{v_1, v_3, v_5, \ldots, v_{2k+1}\}$ is the uniqueγ_{gct}-set of G so that $f_{\gamma gct}(G) = 0$.
Let $n - 1$ be even. Let $n - 1 = 2k$. Then $S_1 = \{v_1, v_3, v_5, \ldots, v_{2k-1}, v_{2k}\}, S_2 = \{v_1, v_3, v_5, \ldots, v_{2k-2}, v_{2k}, v_2\}$are the two γ_{gct}-sets of G such that $f_{\gamma gct}(S_1) = f_{\gamma gct}(S_2) = 1$.so that $f_{\gamma gct}(G) = 1$. ■

Theorem 2.15. For the Banana tree graph $G = B_{r,s}$, $f_{\gamma gct}(G) = 0$.
Proof: Let x be the centre vertex of G and the set of all end vertices of G. Then$S = Z \cup \{x\}$ is the unique γ_{gct}-set of G so that $f_{\gamma gct}(G) = 0$.■

Theorem 2.16. For the sunflower graph $G = SF_n$, $f_{\gamma gct}(G) = 0$.
Proof: Let S be the set of extreme vertices of G. Then S is the unique γ_{gct}-set of G. So that $f_{\gamma gct}(G) = 0$. ■

Theorem 2.17. For every pair$a, b$of integers with $0 \leq a < b, b \geq 2$, there exists a connected graph G such that $f_{\gamma gct}(G) = a$ and $\gamma_{gct}(G) = b$.
Proof: Let $P : u, v, z$be a path of order three. Let $P_i: u_i, v_i$ $(1 \leq i \leq a)$ be a copyof path on two vertices. Let Hbe a graph obtained from $P$and $P_i(1 \leq i \leq a)$ byjoining each u_i $(1 \leq i \leq a)$ with v and each v_i $(1 \leq i \leq a)$ with z. Let G be thegraph obtained from H by introducing new vertices$z_1, z_2, \ldots, z_{b-a+1}$ joining eachz_i $(1 \leq i \leq a)$ with z. The graph G is given in Figure 2.4.

First, we show that $\gamma_{gct}(G) = b$. Let $Z = \{u, z_1, z_2, \ldots, z_{b-a+1}\}$ be the set of endvertices of G. By Theorem 1.1, Z is a subset of every geodetic cototal dominating set of G. Let $H_i = \{u_i, v_i\}$. Then it is easily observed that every geodetic cototaldominating set containing at least one vertex from each $H_i (1 \leq i \leq a)$ and so$\gamma_{gct}(G) \geq b - a + a = b$. Let $S = Z \cup \{u_1, u_2, \ldots, u_a\}$. Then S is a minimum geodeticcototal dominating set of G so that $\gamma_{gct}(G) = b$.

Next, we prove that $f_{\gamma gct}(G) = a$. Since every geodetic co-total dominating set ofGcontains Z, it follows that$f_{\gamma gct}(G) \leq \gamma_{gct}(G) - |Z| = b - (b - a) = a$.Now, since $\gamma_{gct}(G) = b$ and every γ_{gct}-set of G contains Z, it is easily seen that everyγ_{gct}-set of G is of the form $S = Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in H_i$ $(1 \leq i \leq a)$. Let T beany proper subset of S with $|T| < a$. Then there exists an edge e_j $(1 \leq j \leq a)$ suchthat $e_j \notin T$. Let f_j be an edge of H_j distinct from e_j. Then $W_1 = (S - \{e_j\} \cup \{f_j\}$is a γ_{gct}-set properly containing T. Thus W is not the unique γ_{gct}-set containingT. Thus T is not a forcing subset of S. This is true for all minimum geodetic cototaldominating sets of G and so it follows that $f_{\gamma gct}(G) = a$.

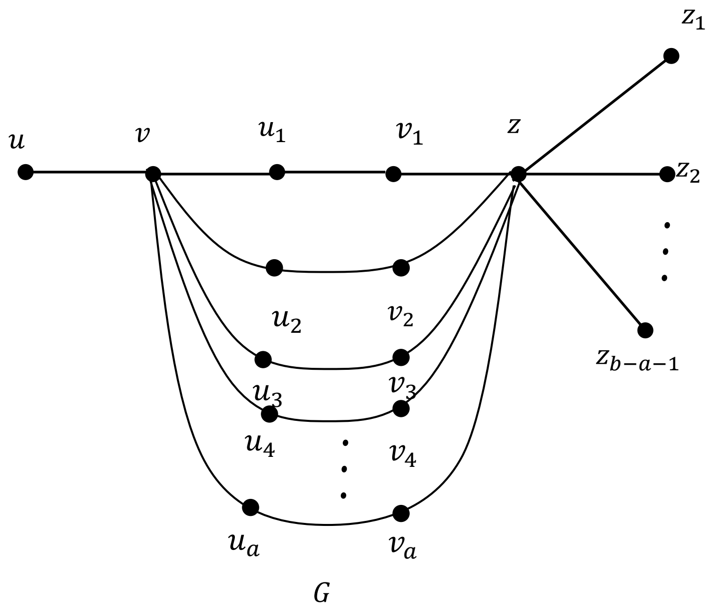


Figure 2.3

3. Conclusion

In this paper we studied the concept of forcing geodetic cototal domination number of some standard graphs some general properties satisfied by this concept are studied. In future studies, the same concept is applied for the other graph operations.

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