Steiner domination decomposition number of graphs

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Abstract

In this paper, we introduce a new concept Steiner domination decomposition number of graphs. Let G be a connected graph with Steiner domination number $\gamma_s(G)$. A decomposition $\pi = \{G_1, G_2, \ldots, G_n\}$ of G is said to be a Steiner Domination Decomposition (SDD) if $\gamma_s(G_i) = \gamma_s(G), 1 \leq i \leq n$. Steiner domination decomposition number of G is the maximum cardinality obtained for an SDD of G and is denoted as $\pi_{std}(G)$. Bounds on $\pi_{std}(G)$ are presented. Also, few characteristics of the subgraphs belonging to SDD of maximum cardinality are discussed.

Keywords: subgraphs; domination; decomposition number.

AMS subject classification: 05C12, 05C69[3]

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1. Introduction

Let G be a simple, connected and undirected graph with vertex set $V(G)$and edge set $E(G)$. The order and size of G are p and q respectively. For standard terminologies and notations, we refer to [1]. Steiner domination number of a graph is a concept introduced by John $et\ al.$ Further studies on this concept is found in [7], [8]. In [5], we introduced the concept of Steiner decomposition number of graphs and in [6] we presented the Steiner decomposition number of Complete $n-$ Sun graph. In this paper, a new decomposition concept called Steiner domination decomposition number of graphs is studied. The following are the basic definitions and results needed for the subsequent section.

Definition 1.1. [2] Let G be a connected graph. For a set $W \subseteq V(G)$, a tree T contained in G is a Steiner tree with respect to W if T is a tree of minimum order with $W \subseteq V(T)$. The set $S(W)$ consists of all vertices in G that lie on some Steiner tree with respect to W. The set W is a Steiner set for G if $S(W) = V(G)$. The minimum cardinality among the Steiner sets of G is the Steiner number $s(G)$.

Definition 1.2. [3] A set $D \subseteq V(G)$in a graph G is called a dominating set if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.

Definition 1.3. [4] For a connected graph $G, W \subseteq V(G)$is called a Steiner dominating set if W is both a Steiner set and a dominating set. The minimum cardinality of a Steiner dominating set of G is said to be Steiner domination number and is denoted by $\gamma_s(G)$. A Steiner dominating set of cardinalities$\gamma_s(G)$ is said to be a $\gamma_s - set$ of G.

Definition 1.4. The decomposition π of a graph G is a collection of edge disjoint subgraphs $G_1, G_2, \dots, G_n$ such that each $G_i, 1 \leq i \leq n$is connected and $E(G) = E(G_1) \cup \mathrm{E}(G_2) \cup \dots \cup \mathrm{E}(G_n)$.

Definition 1.5. [5] For a connected graph G with Steiner number $s(G)$, a decomposition $\pi = \{G_1, G_2, \dots, G_n\}$of G is said to be a Steiner Decomposition(SD) if $s(G_i) = s(G)$ for all $i, (1 \leq i \leq n)$.The maximum cardinality obtained for the Steiner decomposition π of G is called the Steiner decomposition number of G and is denoted by $\pi_{st}(G)$. Steiner decomposition of cardinality $\pi_{st}(G)$ is denoted as SD_{max}.

Theorem 1.6. [4] For any connected graph G of order $p \geq 2$, $\gamma_s(G) = 2$ if and only if there exists a Steiner dominating set $W = \{u, v\}$of G such that $d(u, v) \leq 3$.

Theorem 1.7. [4] For a connected graph G of order $p \geq 2$, $\gamma_s(G) = p$ if and only if $G = K_p$.

Result 1.8. [7] For the path graph on p vertices $(p \geq 2), \gamma_s(P_p) =$

$$\begin{cases} \left\lceil \frac{p-4}{3} \right\rceil + 2 & if\ p \geq 5 \\ 2 & if\ p = 2,3,4 \end{cases}$$

Notation 1.9. $\mathcal{F}_p$ denotes the family of trees of order p with the property that each vertex is either a pendant vertex or a support vertex.

2. Steiner Domination Decomposition

Definition 2.1. A decomposition $\pi = \{G_1, G_2, \dots, G_n\}$of a graph G is called a Steiner Domination Decomposition (SDD) if $\gamma_s(G_i)$=$\gamma_s(G)$, $(1 \leq i \leq n)$. The maximum cardinality obtained for π is called the Steiner domination decomposition number of G and is denoted by $\pi_{std}(G)$. An SDD of cardinality $\pi_{std}(G)$is denoted as SDD_{max}. A graph G with $\pi_{std}(G) = 1$ is said to be non-Steiner domination decomposable graph. If $\pi_{std}(G) \geq 2$ then G is said to be Steiner domination decomposable graph.

Example 2.2. Consider the graph G in figure 1.

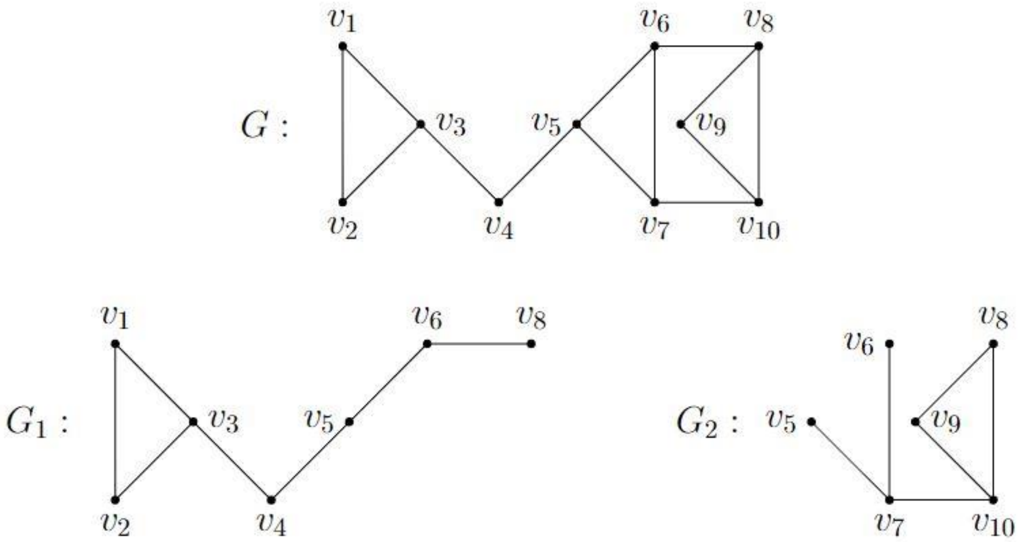

Figure 1. Graph G and its Steiner domination decomposition $\pi = \{G_1, G_2\}$

The set $W = \{v_1, v_2, v_5, v_9\}$is a $\gamma_s - set$ of G. Hence $\gamma_s(G) = 4$. Since $\gamma_s(G_1) = \gamma_s(G_2) = 4 = \gamma_s(G)$, $\pi = \{G_1, G_2\}$ is a SDD. It can be easily verified that π is a SDD_{max}. Thus $\pi_{std}(G) = 2$.

Theorem 2.3. If $\pi_{std}(G) = q$ then $diam\ G < 4$.
Proof. Steiner domination decomposition number of $G, \pi_{std}(G) = q \Leftrightarrow \pi = \{G_i \cong K_2 / 1 \leq i \leq q\}$ is a SDD_{max}. Steiner domination number of K_2 is 2, hence $\gamma_s(G) = 2$. Also, we have $\gamma_s(G) = 2$ implies $diam\ G < 4$. Therefore if $\pi_{std}(G) = q$ then $diam\ G < 4$. Hence proved.

Theorem 2.4. Let G be a connected graph with $\gamma_s(G) \geq 3$. Then $1 \leq \pi_{std}(G) \leq$

$\left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor$.

Proof. From definition 2.1, it is obvious that $\pi_{std}(G) \geq 1$. Let $\pi = \{G_i / 1 \leq i \leq n\}$ be a SDD of G. First to prove $|E(G_i)| \geq \gamma_s(G)$ for all i. Assume to the contrary that $|E(G_i)| < \gamma_s(G)$ for some i. Without loss of generality, let $|E(G_1)| < \gamma_s(G)$. Then $|V(G_1)| \leq \gamma_s(G)$.

Case (i): $|V(G_1)| < \gamma_s(G)$

If $|V(G_1)| < \gamma_s(G)$ then $\gamma_s(G_1) < \gamma_s(G)$. Therefore $G_1 \notin \pi$.

Case (ii): $|V(G_1)| = \gamma_s(G)$

In order to satisfy $\gamma_s(G_1) = \gamma_s(G), G_1$ must be a complete graph on $\gamma_s(G)$vertices. But we have $|V(G_1)| > |E(G_1)|$. Hence G_1 is non isomorphic to $K_{\gamma_s(G)}$. Therefore $G_1 \notin \pi$.

In both the cases, we arrive at a contradiction to our assumption that $G_1 \in \pi$. Hence $|E(G_1)| \geq \gamma_s(G)$. Since G_1 is chosen arbitrarily, we can conclude $|E(G_i)| \geq \gamma_s(G)$ for all i.Thus subgraphs of G belonging to any Steiner domination decomposition should have atleast $\gamma_s(G)$edges and so $\pi_{std}(G) \leq \left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor$. Hence $1 \leq \pi_{std}(G) \leq \left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor$.

Theorem 2.5. Let G be a Steiner domination decomposable graph with q edges. For $\gamma_s(G) = 3, \pi_{std}(G) = \frac{q}{3}$if and only if each $G_i \in SDD_{max}$is isomorphic to either $K_{1,3}$or K_3 and for $\gamma_s(G) > 3, \pi_{std}(G) = \frac{q}{\gamma_s(G)}$ if and only if each $G_i \in SDD_{max}$ is isomorphic to $K_{1,\gamma_s(G)}$.

Proof. Let G be a Steiner domination decomposable graph. Assume $\gamma_s(G) = 3$ and $\pi_{std}(G) = \frac{q}{3}$. Then for any $G_i \in SDD_{max}, |E(G_i)| = 3$ and hence$|V(G_i)| \leq 4$. If $|V(G_i)| \leq 3$ for some i, then the only graph that satisfies $\gamma_s(G_i) = 3$ is K_3. If $|V(G_i)| = 4$ for some i, then G_i is a tree. Star graph $K_{1,3}$ is the unique tree which satisfies the required properties. Thus if $\pi_{std}(G) = \frac{q}{3}$ then $G_i \cong K_{1,3}$or K_3 for all $G_i \in SDD_{max}$. Converse part is obvious. Now, assume$\gamma_s(G) > 3$and $\pi_{std}(G) = \frac{q}{\gamma_s(G)}$.Then$|E(G_i)| = \gamma_s(G)$ for every $G_i \in SDD_{max}$ and so $|V(G_i)| \leq \gamma_s(G) + 1$. There doesn't exist any graph G_i with the properties $|V(G_i)| \leq \gamma_s(G)$ and $\gamma_s(G_i) = \gamma_s(G)$. Since $K_{1,\gamma_s(G)}$ is the unique graph on $\gamma_s(G) + 1$ vertices that has Steiner domination number same as G, we have $|V(G_i)| = \gamma_s(G) + 1$ implies $G_i \cong K_{1,\gamma_s(G)}$. Hence if $\pi_{std}(G) = \frac{q}{\gamma_s(G)}$ then $G_i \cong K_{1,\gamma_s(G)}$ for all $G_i \in SDD_{max}$. Converse is obvious.

Theorem 2.6. Let G be a connected graph with $\gamma_s(G) \geq 3$ and $\left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor = m, (m > 1)$. If $\pi_{std}(G) = m - n, (0 \leq n < m - 1)$ then $\gamma_s(G) \leq |E(G_i)| \leq (n + 2)\gamma_s(G) - 1$ for all $G_i \in SDD_{max}$.

Proof. Let G be a connected graph such that $\gamma_s(G) \geq 3$. Let $\left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor = m, (m > 1)$. Assume $\pi_{std}(G) = m - n$ where $0 \leq n < m - 1$. Let $\pi = \{G_1, G_2, \ldots, G_{m-n}\}$be a SDD_{max} of G. To prove $\gamma_s(G) \leq |E(G_i)| \leq (n + 2)\gamma_s(G) - 1$ for all $G_i \in \pi$. The requirement of edges in each subgraph belonging to any SDD of G is atleast $\gamma_s(G)$. Hence$|E(G_i)| \geq \gamma_s(G)$for every $G_i \in \pi$. Without loss of generality, assume

$|E(G_{m-n})| \geq |E(G_i)|, 1 \leq i \leq m-(n+1)$. Since $\left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor = m$, we get $m\gamma_s(G) \leq q \leq (m+1)\gamma_s(G) - 1$. We know that $\sum_{i=1}^{m-n} |E(G_i)| = q$ and $|E(G_i)| \geq \gamma_s(G)$ for $1 \leq i \leq m-(n+1)$.

Therefore,

$$\sum_{i=1}^{m-n} |E(G_i)| \leq (m+1)\gamma_s(G) - 1$$

$$(m-(n+1))\gamma_s(G) + |E(G_{m-n})| \leq (m+1)\gamma_s(G) - 1$$

$$\Rightarrow |E(G_{m-n})| \leq (n+2)\gamma_s(G) - 1$$

Thus, the possible number of edges in a subgraph belonging to SDD_{max} is atmost $(n+2)\gamma_s(G) - 1$. Hence $\gamma_s(G) \leq |E(G_i)| \leq (n+2)\gamma_s(G) - 1$ for all $G_i \in SDD_{max}$.

Theorem 2.7. Let G be a connected graph with $\gamma_s(G) \geq 5$ and $\left\lfloor \frac{q}{\gamma_s(G)} \right\rfloor = m, (m > 1)$. If $\pi_{std}(G) = m - n, (0 \leq n < m-1)$ then the number of path graphs belonging to SDD_{max} is strictly less than $n+1$.

Proof. Let G be a connected graph with q edges. Let $\gamma_s(G) = k+1$ where $k \geq 4$. Assume $\pi_{std}(G) = m-n, (0 \leq n < m-1)$. Let $\pi = \{G_i / 1 \leq i \leq m-n\}$ be a SDD_{max}. Let N denotes the number of path graphs belonging to π. First we try to prove $N \neq n+1$.

Suppose $N = n+1$. Consider $G_1, G_2, \ldots, G_{n+1} \in \pi$ as path graphs. Path graphs with Steiner domination number $k+1$ are P_{3k-1}, P_{3k} and P_{3k+1}. Therefore $3k-2 \leq |E(G_i)| \leq 3k$ for $1 \leq i \leq n+1$.

Now, $\sum_{i=1}^{m-n} |E(G_i)| = \sum_{i=1}^{n+1} |E(G_i)| + \sum_{i=n+2}^{m-n} |E(G_i)|$

$\geq (n+1)(3k-2) + (m-2n-1)(k+1)$

$$\sum_{i=1}^{m-n} |E(G_i)| \geq (n+2)k - (4n+3) + m(k+1)$$

For $k \geq 4, q \leq (m+1)(k+1) - 1 < (n+2)k - (4n+3) + m(k+1)$. .

This is a contradiction since $\sum_{i=1}^{m-n} |E(G_i)| = q$ and $q \leq (m+1)(k+1) - 1$. Hence $N \neq n+1$. If $N > n+1$ then $\sum_{i=1}^{m-n} |E(G_i)| > (n+2)k - (4n+3) + m(k+1)$. This again results in a contradiction. Thus $N < n+1$ and so number of path graphs belonging to π is strictly less than $n+1$. Hence the proof.

Theorem 2.8. If $T \in \mathcal{F}_p$ then $\pi_{std}(T) = 1$.

Proof. Assume $T \in \mathcal{F}_p$. Every vertex of T is either a pendant vertex or a support vertex. Let l and m be the number of pendant vertices and support vertices of T respectively. Clearly the set of all pendant vertices of T forms the $\gamma_s - set$. Hence $\gamma_s(T) = l$. Number of edges of T is $l + m - 1$. Also, $m \leq l$ for any graph. Hence by theorem 2.4, $\pi_{std}(T) = 1$.

Remark 2.9. If $s(G) = \gamma_s(G)$ then $\pi_{st}(G)$ need not be equal to $\pi_{std}(G)$.

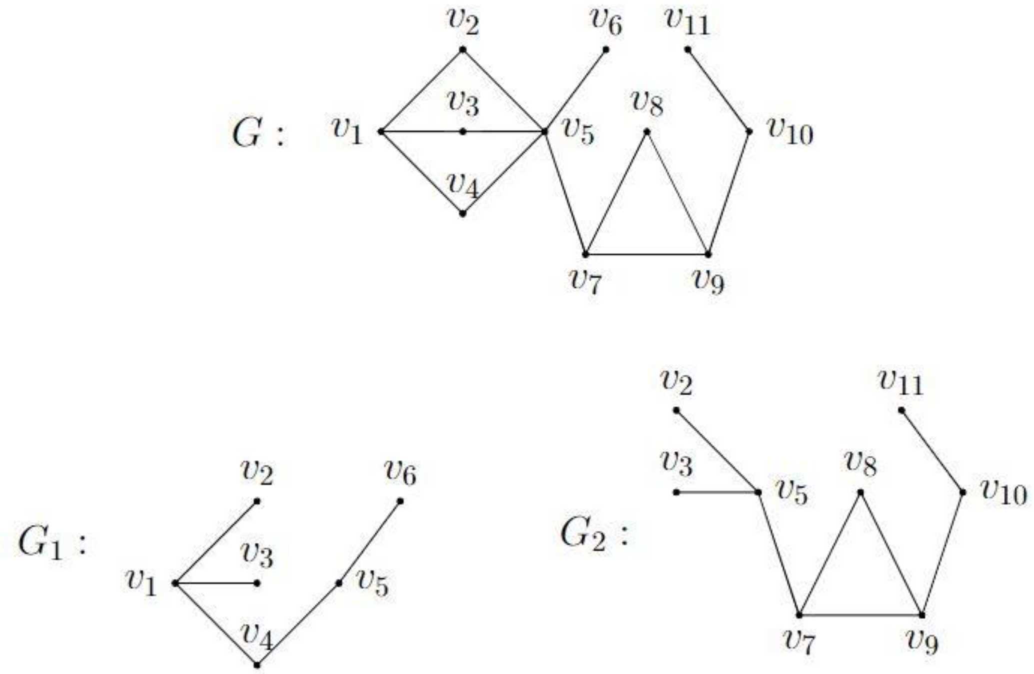

Figure 2. Graph G and its SDD_{max}, $\pi = \{G_1, G_2\}$

For the graph G in figure 2, minimum Steiner set= $\gamma_s - set = \{v_1, v_6, v_8, v_{11}\}$. Hence $s(G) = \gamma_s(G) = 4$. Steiner domination decomposition $\pi = \{G_1, G_2\}$is a SDD_{max} of G and so $\pi_{std}(G) = 2$. Also $\pi_{st}(G) = 1$. Therefore $\pi_{st}(G) \neq \pi_{std}(G)$.

Theorem 2.10. Let G be a connected graph such that $s(G) = \gamma_s(G) = k\ (say)$. If there exist some $SD_{max}$and SDD_{max} for G satisfying the condition that each subgraph in the decompositions is of order $k + 1$ and has a cutvertex of degree k then $\pi_{st}(G) = \pi_{std}(G)$.

Proof. Consider a connected graph G with $s(G) = \gamma_s(G) = k$. Let π_1 and π_2 be the SD_{max} and SDD_{max}respectively which satisfies the condition that each subgraph in both the decompositions is of order $k + 1$ and has a cutvertex of degree k. First to prove, $\pi_1$is a SDD. Let $\pi_1 = \{G_i\ /1 \leq i \leq n\}$. $\pi_1$is a SD implies $s(G_i) = k$ for all i. Each $G_i(1 \leq i \leq n\ )$is of order $k + 1$ and has a cutvertex of degree k.Therefore minimum Steiner set of $G_i = \gamma_s - set$ of G_i for all i and so $\gamma_s(G_i) = k$. Thus π_1 is a SDD. In the similar way, we can prove π_2 is a SD. Now to prove, $\pi_{st}(G) = \pi_{std}(G)$. Suppose $\pi_{st}(G) > \pi_{std}(G)$ then $|\pi_1| > |\pi_2|$. Since π_1 is a SDD, we get a contradiction to π_2 is aSDD_{max}. Suppose $\pi_{st}(G) < \pi_{std}(G)$ then $|\pi_1| < |\pi_2|$. Since π_2 is a SD, we get a contradiction to π_1 is a SD_{max}. Therefore $\pi_{st}(G) = \pi_{std}(G)$.

3. Conclusion

In this paper, we initiate a study on Steiner domination decomposition number of graphs. It is quite interesting to investigate this new parameter and study the properties of the subgraphs belonging to SDD. Future works can be done on calculating the Steiner domination decomposition number for families of graphs and finding the bounds in terms of other graph theoretical parameters.

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