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On The Study of Edge Monophonic Vertex Covering Number

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Abstract

For a connected graph G of order n ≥ 2, a set S of vertices of G is an edge monophonic vertex cover of G if S is both an edge monophonic set and a vertex covering set of G. The minimum cardinality of an edge monophonic vertex cover of G is called the edge monophonic vertex covering number of G and is denoted by $\boldsymbol{m_{e\alpha}(G)}$. Any edge monophonic vertex cover of cardinality $\boldsymbol{m_{e\alpha}(G)}$ is a $\boldsymbol{m_{e\alpha}(G)}$-set of G. Some general properties satisfied by edge monophonic vertex cover are studied.

Keywords: monophonic set; edge monophonic set; vertex coveringset; edgemonophonic vertex cover; edge monophonic vertex covering number.

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1. Introduction

By a graph G=(V,E) , we mean a finite undirected connected graph without loops and multiple edges. The order and size of G are denoted by n and m respectively. Also $\delta(G)$ is the minimum degree in a graph G. For basic graph theoretic terminology, we refer to Harary[7]. The distance d(u,v) between two vertices u and v in a connected graph G is the length of a shortest u-v path in G(1) For a vertex v of G, the eccentricity e(v) is the distance between v and a vertex farthest fromv. The minimum eccentricity among the vertices of G is the radius, rad G and the maximum eccentricity is its diameter, diamG. The neighbourhood of a vertex v of G is the set N(v) consisting of all vertices which are adjacent with v. A vertex v is a simplicial vertex or an extreme vertex of G if the subgraph induced by its neighbourhood N(v) is complete. A caterpillar is a tree of order 3 or more, the removal of whose end vertices produces a path called the spine of the caterpillar.

A diametralpath of a graph is a shortest path whose length is equal to the diameter of the graph. A tree containing exactly two non-pendent vertices is called a doublestar denoted by $S_{K_1 k_2}$ where k_1 and k_2 are the number of pendent vertices on these two non-pendent vertices. A graph G is called triangle free if it does not contain cycles of length 3. A set of vertices no two of which are adjacent is called an independentset. By a matching in a graph G, we mean an independent set of edges of G. A maximalmatching is a matching M of a graph G that is not a subset of any other matching. The independencenumber$\beta(G)$ of G is the maximum number of vertices in an independent set of vertices of G.

A geodeticset of G is a set $S \subseteq V$(G) such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The geodeticnumber$g(G)$ of G is the minimum cardinality of its geodetic sets and any geodetic set of cardinalities$g(G)$ is a minimumgeodeticset or a geodeticbasis or a g-set of G. The geodetic number of a graph was introduced in [2, 8] and further studied in [3-5].

A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called a monophonic path if it is a chordless path. A set S of vertices of G is a monophonic set of G if each vertex v of G lies on an $x-y$ monophonic path for some $x,y \in S$. The minimum cardinality of a monophonic set of G is the monophonicnumber of G and is denoted by $m(G)$. Any monophonic set of cardinalities$m(G)$ is a minimummonophonicsetoramonophonicbasisora$m$-set of G. The monophonic number of a graph was studied and discussed in [9, 12].

A set S of vertices in G is called an edgemonophonicset of G if every edge of G lies on a monophonic path joining some pair of vertices in S and the minimum cardinality of an edge monophonic set is the edgemonophonicnumber$m_e(G)$ of G. An edge monophonic set of cardinalities$m_e(G)$ is called an m_e-set of G. The edge monophonic number of a graph was introduced in and further studied in [10].

A subset $S \subseteq V(G)$ is said to be a *vertexcoveringset* of G if every edge has at least one end vertex in S. A vertex covering set of G with the minimum cardinality is called a *minimumvertexcoveringset* of G. The *vertexcoveringnumber* of G is the cardinality of any minimum vertexcovering set of G. It is denoted by $\alpha(G). The$ vertex covering number was studied in [14].

For a connected graph G of order $n \geq 2$, a set S of vertices of G is an*monophonicvertexcover* of Gif$S$ is both a monophonic set and a vertex covering set of G. The minimum cardinality of ae monophonic vertex cover of G is called the *monophonicvertexcoveringnumber* of G and is denoted by $m_\alpha(G)$. Any monophonic vertex cover of cardinality $m_\alpha(G)$ is a m_α-set of G.

A subset $S \subseteq V(G)$ is a *dominatingset* if every vertex in V-S is adjacent to at least one vertex in S. The minimum cardinality of a dominating set in a graph G is called the *dominating umber* of G and denoted by $\gamma(G)$. The dominating number of a graph was studied in [6]. A set of vertices of G is said to be *monophonicdominationset* if it is both a monophonic set and a dominating set of G. The minimum cardinality of a monophonic domination set of G is called a *monophonicdominationnumber* of G and denoted by $\gamma_m(G)$. The monophonic domination number was studied in [11]. A set of vertices of a graph G is an *edgemonophonicdominationset* if it is both edge monophonic set and a domination set of G. The minimum cardinality of an edge monophonic domination set of G is called an *edgemonophonicdominationnumber* of G and denoted by $\gamma_{me}(G)$. The edge monophonic domination number was studied in [13].

The following theorems will be used in the sequel.

Theorem 1.1. [10] Every extreme vertex of a connected graph G belongs to every edge monophonic set of G.
In particular, each end vertex of G belongs to every edge monophonic set of G.

Theorem 1.2. [10] Let G be a connected graph with cut-vertices and S be an edge monophonic set of G. If v is a cut-vertex of G, then every component of G-v contains an element of S.

2.The Edge Monophonic Vertex Coverofa Graph

Definition 2.1. Let G be a connected graph of order $n \geq 2$. Aset S of vertices of G is an *edgemonophonicvertexcover* of Gif$S$ is both an edge monophonic set and a vertex cover of G. The minimum cardinality of an edge monophonic vertex cover of G is called the *edgemonophonicvertexcoveringnumber* of G and is denoted by $m_{e\alpha}(G)$. Any edge monophonic vertex cover of cardinality $m_{e\alpha}(G)$ is a $m_{e\alpha}$-set of G.

Example 2.2. For the graph G given in Figure 2.1, $S = \{v_1, v_4\}$ is a minimum edge monophonic set of G so that $m_e(G) = 2$ and $S' = \{v_1, v_2, v_4, v_6\}$ is a minimum edge monophonic vertex cover of G so that $m_{e\alpha}(G) = 4$. Thus, the edge monophonic number and the edge monophonic vertex covering number of a graph are different.

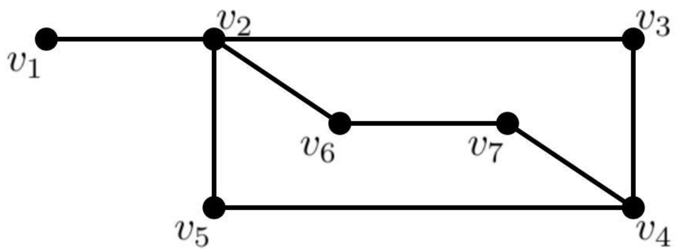

Figure 2.1: G

Remark 2.3. For the graph G given in Figure 2.2, $S = \{v_1, v_2, v_5\}$ is a minimum monophonic vertex cover of G so that $m_\alpha(G) = 3$ and $S' = \{v_1, v_2, v_3, v_4\}$ is a minimum edge monophonic vertex cover of G so that$m_{e\alpha}(G) = 4$. Hence the monophonic vertex covering number is different from the edge monophonic vertex covering number of a graph.

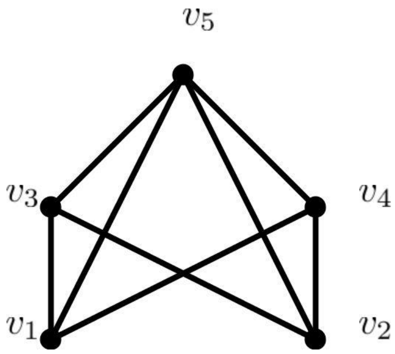

Figure 2.2: G

Remark 2.4. For the graph G given in Figure 2.3, $S = \{v_1, v_{11}\}$ is a minimum edge monophonic set of G so that $m_e(G) = 2.S' = \{v_1, v_8, v_{11}\}$ is a minimum edge monophonic dominating set of G so that $\gamma_{me}(G) = 3$ and $S'' = \{v_1, v_2, v_3, v_4, v_8, v_{11}\}$ is a minimum edge monophonic vertex cover of G so that $m_{e\alpha}(G) = 6$. Hence the edge monophonic vertex covering number of a graph is different from the edge monophonic number and the edge monophonic dominating number of a graph.

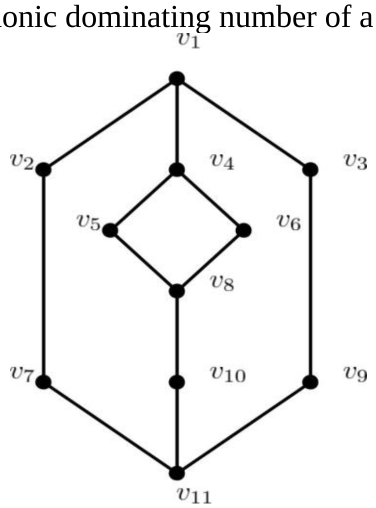

Figure 2.3: G

Theorem 2.5. For any connected graph G,$2 \leq \max\{\alpha(G), m_e(G)\} \leq m_{e\alpha}(G) \leq n$.

Proof. Any edge monophonic set of G needs at least 2 vertices and so $2 \leq \max\{\alpha(G), m_e(G)\}$. From the definition of edge monophonic vertex cover of G, we have, $\max\{\alpha(G), m_e(G)\} \leq m_{e\alpha}(G)$. Clearly V(G) is an edge monophonic vertex cover of G. Hence $m_{e\alpha}(G) \leq n$. Thus $2 \leq \max\{\alpha(G), m_e(G)\} \leq m_{e\alpha}(G) \leq n$.□

Remark 2.6. The bounds in Theorem 2.5 are sharp. For the complete graph $K_n (n \geq 2)$,$m_{e\alpha}(K_n) = n$. In Remark 2.3, we have, $m_e(G) = m_{e\alpha}(G) = 4$
The bounds are strict in Example 2.2 as $\alpha(G) = 3$,$m_e(G) = 2$,$m_{e\alpha}(G) =4$.Here $2< 3 < 4 < 7$.

Remark 2.7. Clearly union of a vertex covering set and an edge monophonic set of G is an edge monophonic vertex cover of G. In Figure 2.1, $S = \{v_1, v_2, v_4, v_6\}$ is an edge monophonic vertex cover, in Figure 2.2, $S= \{v_1, v_2, v_3, v_4, v_5\}$ is an edge monophonic vertex cover and in Figure 2.3, $S = \{v_1, v_2, v_3, v_4, v_8, v_{11}\}$ is an edge monophonic vertex cover.
Thus $2 \leq \max\{\alpha(G), m_e(G)\} \leq m_{e\alpha}(G) \leq \min\{\alpha(G) + m_e(G), n\}$.

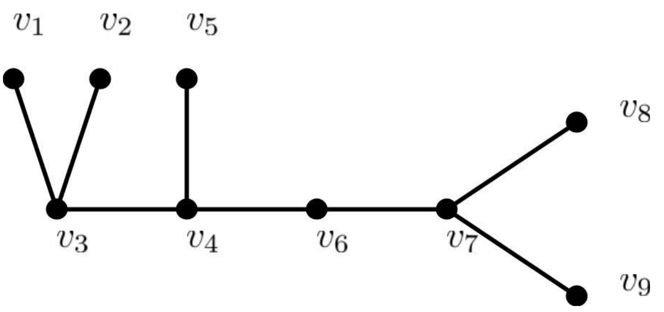

Figure 2.4: G

For the graph G in Figure 2.4, we observe that $S_1 = \{v_3, v_4, v_7\}$ is a minimum vertex cover of G so that $\alpha(G) = 3$,$S_2 = \{v_1, v_2, v_5, v_8, v_9\}$ is a minimum edge monophonic set of G so that $m_e(G) = 5$ and $S_3 =\{v_1, v_2, v_3, v_4, v_5, v_7, v_8, v_9\} = S_1 \cup S_2$ is a $m_{e\alpha}$-set of G and so $m_{e\alpha}(G) =8 = \alpha(G) + m_e(G) < n = 9$.

Theorem 2.8. Each extreme vertex of G belongs to every edge monophonic vertex cover of G.
In particular, each end vertex of G belongs to every edge monophonic vertex cover of G.

Proof. From the definition of $m_{e\alpha}$-set, every $m_{e\alpha}$-set of G is a m_e-set of G. Hence the result follows from Theorem 1.1. □

Corollary 2.9. For any graph G with k extreme vertices, $max\{2, k\} \leq m_{e\alpha}(G) \leq n$.

Proof. The result follows from Theorem 2.5 and Theorem 2.8. □

Corollary 2.10. Let $K_{1,n-1} (n \geq 3)$ be a star. Then $m_{e\alpha}(K_{1,n-1}) = n - 1$.

Proof. Let x be the centre and $S = \{v_1, v_2, \ldots, v_{n-1}\}$ be the set of all extreme vertices of $K_{1,n-1}(n \geq 3)$. Clearly S is a minimum edge monophonic vertex cover of $K_{1,n-1}(n \geq 3)$ by Theorem 2.8. Hence $m_{e\alpha}(K_{1,n-1}) = n - 1$.□

Corollary 2.11. For the complete graph $K_n(n \geq 2)$,$m_{e\alpha}(K_n) = n$.
Proof. Since every vertex of the complete graph $K_n(n \geq 2)$ is an extreme vertex, by Theorem 2.8, the vertex set is the unique edge monophonic vertex cover of K_n. Thus $m_{e\alpha}(K_n) = n$.□

Remark 2.12. The converse of Corollary 2.11 need not be true.
For the graph G given in Figure 2.5, $S = \{v_1, v_2, v_3, v_4, v_5\}$ is an $m_{e\alpha}$-set of G so that $m_{e\alpha}(G) = 5$ and G is not complete.

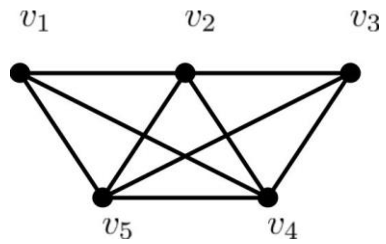

Figure 2.5: G

Theorem 2.13. Let G be a connected graph with cut-vertices and S be an edge monophonic vertex cover of G. If v is a cut-vertex of G, then every component of G-v contains an element of S.
Proof. From the definition of $m_{e\alpha}$-set, every $m_{e\alpha}$-set of G is a m_e-set of G. Hence the result follows from Theorem 1.2. □

Remark 2.14. The cut-vertex of G in Theorem 2.13 need not belong to S. For the graph G given in Figure 2.6, $S = \{v_1, v_3, v_5, v_6\}$ is aa edge monophonic vertex cover of G. Here 4 is a cut-vertex which does not belong to S and 3 is a cut-vertex which belong to S.

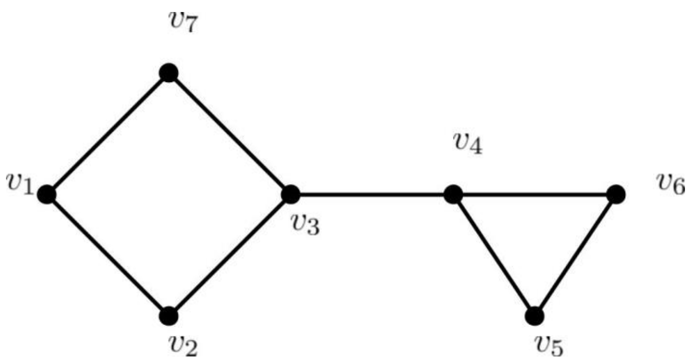

Figure 2.6: G

Theorem 2.15. If a and n are positive integers such that $2 \leq a \leq n$, then there exists a connected graph G of order n with $m_{e\alpha}(G) = a$.

Proof. We prove this theorem by considering two cases.

Case (i): $2 \leq a = n$. Let $G = K_n$. Then by Theorem 2.11, $m_{e\alpha}(G) = n = a$.

Case (ii): $2 \leq a < n$. Consider $H = K_{a-1}$, the complete graph on a-l vertices u_1,u_2,…, u_{a-1}. Add $n - a + 1$ new vertices v_1,v_2,... v_{n-a},x to H by joining the vertices v_1,v_2,…,v_{n-a} to both u_{a-1} and $x$and the graph G is shown in Figure 2.7.

Let $S = \{u_1, u_2, \dots, u_{a-2}\}$ be the set of all extreme vertices of G. Then by Theorem 1.1, they must belong to every edge monophonic set. Also, we observe that $S' = S \cup \{x\}$ is a minimum edge monophonic set. Also the edges of K_{a-1} and the edges $v_i x (1 \leq i \leq n - a)$ are covered by the vertices of S'. Now to cover the edges $u_{a-1}v_i(1 \leq i \leq n - a)$, we must include at least the vertex u_{a-1} to S'. Hence $S'' = \{u_1, u_2, \dots, u_{a-2}, u_{a-1}, x\}$ is a minimum edge monophonic vertex cover of G. Thus $m_{e\alpha}(G) = a < n$.□

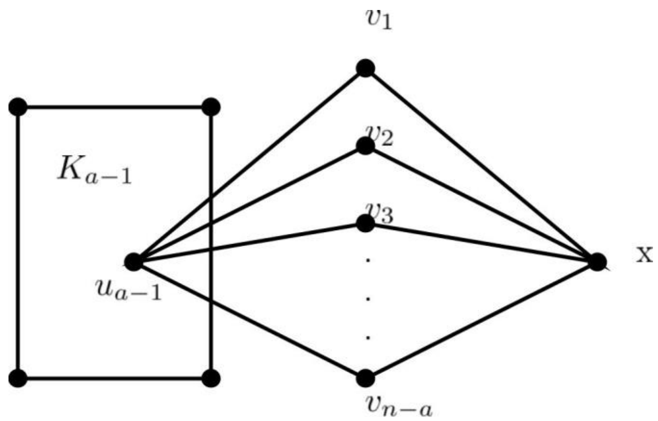

Figure 2.7: G

3.Conclusions

In this paper we analysed the edge monophonic vertex covering number of a graph. It is more interesting to continue my research in this area and it is very useful for further research.

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