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The Detour Domination and Connected Detour Domination values of a graph

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Abstract

The number of $\boldsymbol{\gamma dn}$-sets that $\boldsymbol{v}$ belongs to in G is defined as the detour domination value of $\boldsymbol{v}$, indicated by$\boldsymbol{\gamma D_V(v)}$, for each vertex $\boldsymbol{v \in V(G)}$. In this article, we examined at the concept of a graph's detour domination value. The connected detour domination values of a vertex $\boldsymbol{v \in V(G)}$, represented as$\boldsymbol{CD_V(G)}$, are defined as the number of $\boldsymbol{Cdn}$-sets to which a vertex belongs $\boldsymbol{v}$ to G. Some of the related detour dominating values in graphs' general characteristics are examined. This concept's satisfaction of some general properties is investigated. Some common graphs are established.

Keywords: domination number; detour number; detour domination value; connected detour domination value; etc.

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1. Introduction

Graph having the type G = (V, E) is a finite, undirected connected graph without loops or numerous edges. The order and size of the letter G are represented by the characters n and m, respectively. We refer to [3] for the fundamental terms used in graph theory. If uv is an edge of G, then two vertices u and v are said to be adjacent. If two edges of G connect a vertex, they are said to be adjacent. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is thelength of a shortest u-$v$path in G. A u-v geodesic is a u-v path of length $d(u, v)$.

The longest u-v path in G is also referred to as *detour distance* $D(u, v)$ between two vertices u and v in a linked graph G from u to v. A u-v detour is a u-v path of length $D(u, v)$. If x is a vertex of P that also contains the vertices u and v, then x is said to lie on a u-v detour. Every vertex of G is contained in a detour connecting some pair of vertices in S, which is the definition of a detour set of G. Any detour set of orderdn (G) is referred to as a minimum *detour set* of G or a dn -set of G. The *detour number* dn (G) of G is the minimum order of a detour set. These ideas have been researched in [4, 5, 6]. If for every v ∈ V \ D is adjacent to a vertex in D, then the set D⊆ V is a dominant set of G. If no subset of a dominating set D is a dominating set of G's, then D is said to be minimal. The symbol$\gamma(G)$ denotes the domination number of G, which is the least cardinality of a minimal set of G dominating sets. In [4], the graph's domination number was studied. If a set S is both a detour and a dominating set of G's, then it is referred to as a detour dominating set of G. Any detour dominating set of order $\gamma_d(G)$is referred to as a γ_d- set of G. The *detour domination number*$\gamma_d(G)$ of G is the minimal order of its detour dominating set. In [8], the detour domination number of a graph $\gamma_d(G)$studied. If a set S is a detour dominating set of G and its induction by S is connected, the set S ⊆ V(G) is referred to as a *connected detour dominating set* of G . Any connected detour dominating set with order $\gamma_{cd}(G)$ is referred to as aγ_{cd} - set of G. The *connected detour domination number* of $\gamma_{cd}(G)$ of G is the maximum order of its connected detour dominating sets. In [8,9], the connected detour domination number of a graph was investigated. The subsequent theorem is applied thereafter.

Theorem 1.1[3] Every detour set of a connected graph G contains each end vertex.

Theorem 1.2[3] Let G be a connected graph $n \geq 2$. Then $dn(G) = n$ if and only if $G = K_2$.

Theorem 1.3[3] Let G be a connected graph of order $n \geq 4$. Then $dn(G) = n - 1$ if and only if $G = K_{1,n-1}$.

2.The Detour Domination Value of a Graph

Definition 2.1. For each vertex$v \in V(G)$, we define the detour domination value of v, denoted by $\gamma D_V(v)$, to be the number of γdn-sets to which v belongs to G.

Example 2.2. In relation to the graph G in Figure 2.1, S_1= $\{v_1, v_3\}$,S_2= $\{v_1, v_2\}$, S_3= $\{v_1, v_4\}$, S_4= $\{v_2, v_3\}$, S_5= $\{v_2, v_4\}$, S_6= $\{v_3, v_4\}$are the onlysix minimum γdn-sets of G such that $\gamma D_V(v_1) = 3, \gamma D_V(v_2) = 3, \gamma D_V(v_3) = 3, \gamma D_V(v_4) = 3,\ \gamma\tau(G) = 6$.

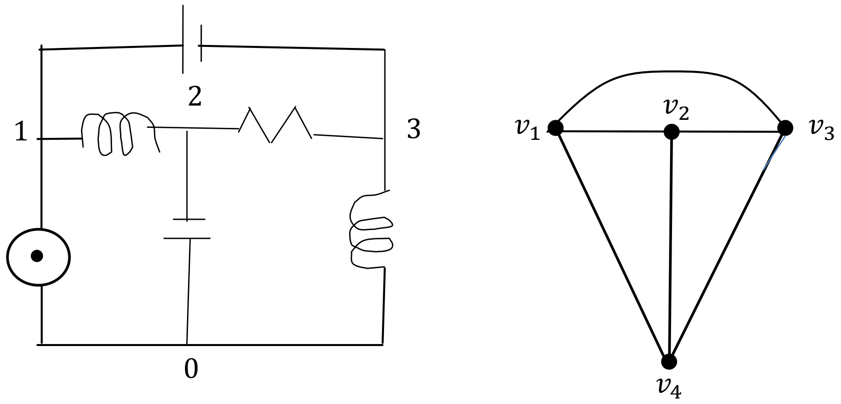

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Figure 2.1

Theorem 2.3. For the complete graph$G$= $K_n (n \geq 2) \gamma D_V(v) = n - 1,\ \gamma\tau(G) = nC_2$for each $v \in V(G)$.

Proof. Since any two sets of G's vertices is the γdn-set of G, thus $\gamma\tau(G) = nC_2$. Since each vertex of G belongs to exactly $n - 1 \gamma dn$-sets, it follows that $\gamma D_V(v) = n - 1$ for each $v \in V(G)$.■

Theorem 2.4. For a star G= $K_{1,n-1} (n \geq 3)\ \gamma D_V(v) = 1, \gamma\tau(G) = 1$ for each $v \in V(G)$.

Proof. We have $G = K_{1,n-1}$. Let S represent the set of all of the end vertices in G. Then S is the unique γdn-set of G. Thus $\gamma\tau(G) = 1$. Therefore $\gamma D_V(v) = 1$ for each $v \in V(G)$.■

Theorem 2.5. For the complete bipartite graph $G = K_{m,n}$with bipartite sets X and Y. and $\gamma\tau(G) = \begin{cases} mn & if\ m, n \geq 2 \\ 1 & if\ m = n = 1 \\ 1 & if\ \{m, n\} = \{1, x\} where\ x > 1 \end{cases}$

If $m, n \geq 2$ then $\gamma D_V(v) = \begin{cases} n, & if\ v \in X \\ m, & if\ v \in Y \end{cases}$

If $m = n = 1 = \gamma D_V(v) = 1$ for any v in $K_{1,1}$.

If $\{m,n\} = \{1,x\}$ with $x > 1$ then$G = K_{1,x}.\gamma D_V(v) = \begin{cases} 1, & if\ v\ \in X \\ 0, & if\ \ v\ \in Y \end{cases}$

Proof. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the two bipartite sets of G. Since any two adjacent vertices of G is a γdn-sets of G, it follows that $\gamma\tau(G) = mn$ if $m,n \geq 2$.
If $m,n = 1$ then it has only one a γdnset of G such that $\gamma\tau(G) = 1$.
If $\{m,n\} = \{1,x\}$ then it only one γdn-set of G such that $\gamma\tau(G) = 1$.
If $v\ \in X$ then any vertex in Y belongs to aγdn-set of G hence $\gamma D_V(v) = n$. Also if $v\ \in Y$,then any vertex in X belongs to a γdn-set thus $\gamma D_V(v) = m$ for $m,n \geq 2$. If $m = n = 1$, then $G = K_2, \gamma D_V(v) = 1$ for any v in $K_{1,1}$. If $\{m,n\} = \{1,x\}$ with $x > 1$, then$G = K_{1,x}, \gamma D_V(v) = \begin{cases} 1, & if\ v\ \in X \\ 0, & if\ \ v\ \in Y \end{cases}$.■

Theorem 2.6. For the wheel graph $G = K_1 + C_{n-1}\ (n \geq 5)$, $\gamma\tau(G) = \begin{cases} 10, & n = 5 \\ 2n-2, & n \geq 6 \end{cases}$ and $\gamma D_V(v) = \begin{cases} 4, & if\ n = 5 \\ 3, & if\ n\ \geq 6\ \text{ and } v \in V(C_{n-1}) \\ n-1, & if\ n \geq 5\ , v \in V(K_1) \end{cases}$

Proof. Let $V(K_1) = x$ and $V(C_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$. Let $n = 5$. Then $S_1 = \{v_1, v_2\}$, $S_2 = \{v_2, v_3\}$, $S_3 = \{v_3, v_4\}$,$S_4 = \{v_4, v_1\}$, $S_5 = \{v_1, x\}$, $S_6 = \{v_2, x\}$, $S_7 = \{v_3, x\}$, $S_8 = \{v_4, x\}$, $S_9 = \{v_2, v_4\}$, $S_{10} = \{v_1, v_3\}$ are γdn-sets of G such that $\gamma D_V(v_1) = 4, \gamma D_V(v_2) = 4, \gamma D_V(v_3) = 4, \gamma D_V(v_4) = 4,\ \gamma D_V(x) = 4$.
Let $n \geq 6$. Then any two adjacent vertices of G is a γdn-set of G so that $\gamma\tau(G) = (n-1) + (n-1) = 2n-2$ for $v \in V(C_{n-1})$, hence v lies in exactly three γdn-set of G so that $\gamma D_V(v) = 3$ for all $v \in V(C_{n-1})$. Since x is adjacent to $n-1$ vertices of $G, \gamma D_V(x) = n-1$.■

Theorem 2.7. For the cycle graph $G = C_n (n \geq 3)$, $\gamma\tau(C_n) = \begin{cases} 3 & if\ n \equiv 0 (mod\ 3) \\ n\left(1 + \frac{1}{2}\left\lfloor\frac{n}{3}\right\rfloor\right) & if\ n \equiv 1 (mod\ 3) \\ n & if\ n\ \equiv 2 (mod\ 3) \end{cases}$

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Let $n = 3k$, where $k \geq 1$, Here $\gamma dn(C_n) = k$,a γdn-set $\mathcal{F}$ comprises $k\ K_1$'s and $\mathcal{F}$ is fixed by the choice of the first K_1. There exists exactly one $\gamma dn(C_n)$-set containing the vertex v_1, and there are two $\gamma dn(C_n)$-sets omitting the vertex v_1 such as $\mathcal{F}$ containing the vertex v_2 and $\mathcal{F}$ containing the vertex v_n. Thus $\gamma\tau(C_n) = 3$.
Let $n = 3k + 1$, where $k \geq 1$. Here $\gamma dn(C_n) = k + 1$, aγdn-set $\mathcal{F}$is constituted in exactly one of the following two ways.
i) $\mathcal{F}$comprises $(k-1)K_1$'s and one K_2.
ii) $\mathcal{F}$comprises$(k+1)K_1$'s.

Case(i) $\langle \mathcal{F} \rangle \cong (k-1)K_1 \cup K_2$: Note that $\mathcal{F}$ is fixed by the choice of the single K_2 choosing a K_2 in the same as choosing its initial vertex in the counter clockwise order. Hence $\tau = 3k+1$.
Case(ii)$\langle \mathcal{F} \rangle \cong (k+1)K_1$:It is clear that each$K_1$ dominates three vertices, exactly there are two vertices, say x and y, each of whom is adjacent to two distinct K_1's in $\mathcal{F}$. And $\mathcal{F}$ is fixed by the placements of x and y.There are $n = 3k+1$ ways of choosing x. Consider the P_{3k-2} (a sequence of $3k-2$ slots) obtained as a result of cutting from C_n the P_3 centered about x vertex. y may be placed in the first slot of any of the $\left\lceil \frac{3k-2}{3} \right\rceil = k$. As the order of selecting the two vertices x and y is immaterial $\tau = \frac{(3k+1)}{2}k$.
Summing over the two disjoint cases, we get $\gamma\tau(C_n) = (3k+1) + \frac{(3k+1)}{2}k = (3k+1)\left(1+\frac{k}{2}\right) = n\left(1+\frac{1}{2}\left\lfloor \frac{n}{3} \right\rfloor\right)$
Let $n = 3k+2$, where $k \geq 1$, Here$\gamma dn(C_n) = k+1$, a $\gamma dn(C_n)$-set $\mathcal{F}$ comprises of only $K_1's$ and is fixed by the placement of the only vertex which is adjacent to two distinct $K_1's$in $\mathcal{F}$. Hence $\gamma\tau(C_n) = n$.■

3. The Connected Detour Domination Value of a Graph

Definition 3.1. For each vertex $v \in V(G)$,we define the connected detour domination values of v, denoted by $CD_V(G)$to be the number of Cdn-sets to which v belongs to G.

Example 3.2. For the graph G given in Figure 3.1, $S_1 = \{v_1, v_2\}$,$S_2 = \{v_1, v_3\}$,$S_3 = \{v_1, v_4\}$, $S_4 = \{v_2, v_4\}$,$S_5 = \{v_2, v_3\}$, $S_6 = \{v_2, v_5\}$, $S_7 = \{v_3, v_5\}$, $S_8 = \{v_4, v_5\}$are the only eight minimum Cdn-sets of G such that $CD_V(v_1) = 3, CD_V(v_2) = 4, CD_V(v_3) = 3$, $CD_V(v_4) = 3, CD_V(v_5) = 3and\tau_c(G) = 8$.

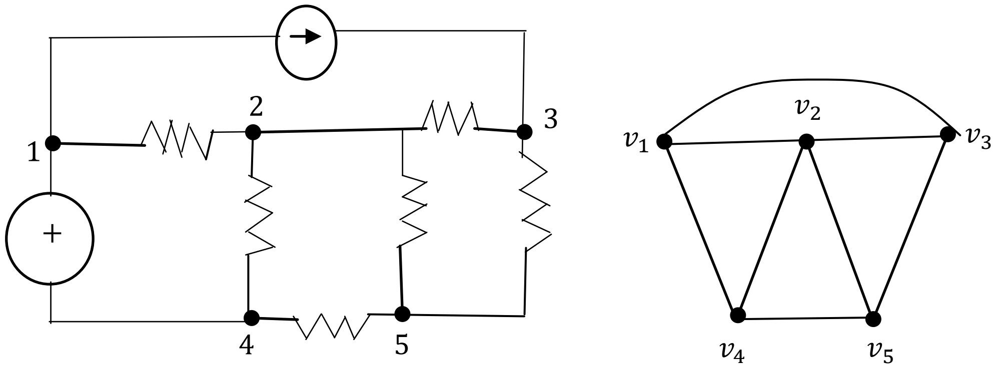

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Figure 3.1

Proposition 3.3. Let G be a graph with n vertices without cut vertices and $\Delta = n-1$. Then $Cdn(G) = 2$ and $CD_V(v) \le n-1 \ \forall v \in V(G)$ and equality holds if and only if $\deg(v) = n-1$.

Proof. Let x be a universal vertex of G. Let $y \in N(x)$. Then $S = \{x, y\}$ is a Cdn-set of G so that $Cdn(G) = 2$. Since x is a universal vertex of G x belongs to every Cdn-set of G. Since G contains at most $n-1$ Cdn-sets, $CD_V(v) \le n-1$. Let $CD_V(v) = n-1$. Hence it follows that v belongs to every Cdn-set of G. Therefore $CD_V(v) = n-1$. The converse is clear. ■

Theorem3.4. For $n \ge 3, \tau_c(C_n) = n$ and $CD_V(v) = n-2 \ \forall v \in V(G)$.

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Then $S_i = V(C_n) - \{v_i, v_{i+1}\} (1 \le i \le n-1)$ and $S = V(C_n) - \{v_1, v_n\}$ are the n, Cdn-sets of G, so that $\tau_c(C_n) = n$. As C_n is vertex transitive $CD_V(v) = CD_V(v_1)$ for all $v \in V(C_n)$. Since v_1 belongs to $n-2$ Cdn-sets of C_n, it follows that $CD_V(v) = n-2$ for all $v \in V(C_n)$. ■

Theorem3.5. For $n \ge 2, \tau_c(P_n) = 1$ and $CD_V(v) = 1$ for each vertex $\forall v \in V(P_n)$.

Proof. Since $S = V(G)$ is the unique Cdn-sets of G the results follow theorem. ■

Theorem3.6. For the complete graph $G = K_n (n \ge 4), CD_V(v) = n-1, \tau_c(G) = nC_2$ for each vertex $v \in V(G)$.

Proof. Since any two set of vertices of G is the $\underline{Cdn}$-set of G, it follows that $\tau_c(G) = nC_2$. Since each vertex of G belongs to exatly $n-1$ Cdn-sets, it follows that $CD_V(v) = n-1$, for each vertex $v \in V(G)$. ■

Theorem3.7. For the wheel graph $G = K_1 + C_{n-1} (n \ge 5), \tau_c(G) = \begin{cases} 10, & n \ge 5 \\ 2n-2, & n \ge 6 \end{cases}$ and $CD_V(v) = \begin{cases} 4, & if\ n = 5 \\ 3, & if\ n \ge 6\ and\ v \in V(C_{n-1}). \\ n-1, & if\ n \ge 5,\ v \in V(K_1) \end{cases}$

Proof. Let $V(K_1) = x$ and $V(C_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$. Let $n = 5$. Then $S_1 = V(v_1, v_2), S_2 = V(v_2, v_3), S_3 = V(v_3, v_4), S_4 = V(v_4, v_1), S_5 = V(v_1, x), S_6 = V(v_2, x), S_7 = V(v_3, x), S_8 = V(v_4, x), S_9 = V(v_2, v_4), S_{10} = V(v_1, v_3)$ are the Cdn-sets of G, such that $CD_V(v_1) = 4, CD_V(v_2) = 4, CD_V(v_3) = 4, CD_V(v_4) = 4, CD_V(x) = 4$ and $\tau_c(G) = 10$. Let $n \ge 6$. Then any two adjacent vertices of G is a Cdn-sets of G so that $\tau_c(G) = (n-1) + (n-1) = 2n-2$ for $v \in V(C_{n-1}), v$ lies in excatly three Cdn-sets of G so that $CD_V(v) = 3$ for all $v \in V(C_{n-1})$. Since x is adjacent to $n-1$ vertices of $G, CD_V(x) = n-1$. ■

Theorem3.8. Let $G = K_1 + P_{n-1}$ and $V(K_1) = x$ and $V(P_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$.

Then for $n-1$ is odd $\tau_c(G)=3$ and $CD_V(v)=\begin{cases} 2, & if\ v=x,v_1,v_{n-1} \\ 0, & otherwise \end{cases}$ and for $n-1$ is even $\tau_c(G)=4$ and $CD_V(v)\begin{cases} 3, & if\ v=x \\ 2, & if\ v=v_1\ or\ v_{n-1} \\ 1, & if\ v=v_{\frac{n-1}{2}} \\ 0, & otherwise \end{cases}$.

Proof. Let $V(K_1)=x$ and $V(P_{n-1})=\{v_1,v_2,\dots,v_{n-1}\}$.

Case (i) $n-1$ is odd. $S_1=\{x,v_1\}, S_2=\{x,v_{n-1}\}, S_3=\{v_1,v_{n-1}\}$ are the only three Cdn-sets of G, such that $CD_V(x)=2, CD_V(v_1)=2, CD_V(v_{n-1})=2$ so that $\tau_c(G)=3$.

Case (ii) $n-1$ is even. $M_1=\{x,v_1\}, M_2=\{x,v_{n-1}\}, M_3=\left\{x,v_{\frac{n-1}{2}}\right\}, M_4=\{v_1,v_{n-1}\}$ arethe only four Cdn-sets of G, so that $CD_V(x)=3, CD_V(v_1)=2, CD_V(v_{n-1})=2$, $CD_V\left(v_{\frac{n-1}{2}}\right)=1$ and$\tau_c(G)=4$. ■

Theorem3.9. $\tau_c(P_2\times P_n)=\begin{cases} 4, & if\ n=2 \\ 1, & if\ n=3 \\ 8, & if\ n\geq 4 \end{cases}$

Proof. Let S be a Cdn-sets of $P_2\times P_n$ of cardinality n where $n\geq 2$ if $n=2$, then $P_2\times P_n\cong C_4$ and any two adjacent vertices form a Cdn-set i.e.$\{u_1,v_1\}$,$\{u_1,u_2\}$, $\{v_1,v_2\}$, $\{u_2,v_2\}$are all possible<u>Cdn</u>-sets of $P_2\times P_2$. If $n=3$, there is a unique<u>Cdn</u>-set$\{u_2,v_2\}$. So let $n\geq 4$. By lemma 2.2 either $\{u_3,u_4,\dots,u_{n-3},u_{n-2}\}\subset S$ or $\{v_3,v_4,\dots,v_{n-3},v_{n-2}\}\subset S$ (and not both). Let $\{u_3,u_4,\dots,u_{n-3},u_{n-2}\}\subset S$. As $v_3\notin S$, to maintain connectedness of $\langle S\rangle$ and to dominate u_1, we have $u_2\in S$. In the same way, $u_{n-1}\in S$. Thus $\{u_2,u_3,\dots,u_{n-2},u_{n-1}\}\subset S$. Since S contains n elements, let the other 2 vertices in S be l,m. To dominate u_1 and v_1, one of l and m (say l) must be either $u_1$or v_2. Similarly m is either u_n or v_{n-1}.Since there are two choices each for l and m such that S forms a Cdn-set, the number of Cdn-sets containing$u_3,u_4,\dots,u_{n-3},u_{n-2}$ is 4. Similarlythe number of Cdn-sets containing$v_3,v_4,\dots,v_{n-3},v_{n-2}$ is 4. Hence by lemma 2.2, we get $\tau_c(P_2\times P_n)=8$ for $n\geq 4$.■

Theorem3.10. Let$P_2\times P_n$ be a rectangular grid with $n\geq 2$ and let $a_i=u_i$ or v_i. If $n=2$, then $CD_V(v)=2$ for all $v\in V(P_2\times P_n)$. If $n=3$, then $CD_V(a_1)=CD_V(a_3)=0$ and $CD_V(a_2)=1$, If $n\geq 4$ then $CD_V(a_i)=\begin{cases} 2, & if\ i=1\ or\ n \\ 6, & if\ i=2\ or\ n-1 \\ 4, & otherwise \end{cases}$

Proof. The proof is clear for $n=2$ and theorem 2.10, so we assume that $n\geq 4$. Let v be a vertex in $P_2\times P_n$.

Case 1: $[v\in\{u_1,v_1,u_n,v_n\}]$. Let $v=u_1$, then using the line of proof of Theorem 3.10, theCdn-sets containing u_1 are precisely those where $l=u_1$ and m is either u_n or v_{n-1} i.e,$CD_V(v)=2$. Same for the case when $v=V_1$ or $v=u_n$ or $v=v_n$.

Case 2: $[v \in \{u_2, v_2, u_{n-1}, v_{n-1}\}]$. Let $v = u_2$. Note that any connected dominating set contains either u_2, v_2. Also total number of minimum connected dominating sets is 8, out of which only two does not contain u_2, namely $\{v_1, v_2, \dots, v_n\}$ and $\{v_1, v_2, \dots, v_{n-1}, u_{n-1}\}$. Thus $CD_V(u_2) = 8 - 2 = 6$. Now,as there exist isomorphisms which maps u_2 to v_2, u_{n-1}, v_{n-1}respectively, by proposition 2.2, we have $CD_V(u_2) = CD_V(v_2) = CD_V(u_{n-1}) = CD_V(v_{n-1}) = 6$.
Case 3: $[v \notin \{u_1, v_1, u_2, v_2, u_{n-1}, v_{n-1}, u_n, v_n\}]$. In this case, from the proof of Theorem 2.10 we have $CD_V(v) = 4$.

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