Outer independent square free detour number of a graph

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Abstract

For a connected graph $G=(V,E)$, a set S of vertices is called an outer independent square free detour set if S is a square free detour set of G such that either $V=S^*$ or $V-S^*$ is an independent set. The minimum cardinality of an outer independent square free detour set of G is called an outer independent square free detour number of G and is denoted by $dn_{\square f}^{oi}(G)$. We determine the outer independent square free detour number of some graphs. We characterize the graph which realizes the result that for any pair of integers α and β with $2 \leq \alpha \leq \beta$, there exists a connected graph G of order $\beta+3$ with square free detour number α and outer independent square free detour number β.

Keywords: square free detour set; outer independent square free detour set; outer independent square free detour number.

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1. Introduction

In this article, a graph G is considered to be a finite, undirected and connected graph of order $n(n \geq 2)$ with neither loops nor multiple edges. Let $D(u, v)$ be the longest path in G and a $u - v$ path of $D(u, v)$ is called $u - v$ detour. The parameters on detour concept were developed by Chartrand [2]. The detour concept was extended to triangle free detour concept by S. Athisayanathan et al. [1, 7]. The detour concept was applied in domination by number of authors. The detour domination number was studied and extended to outer independent detour domination by number of authors in [5, 6].

For any two vertices u, v in a connected graph G, the $u - v$ path P is called $u - v$ triangle free path if no three vertices of P induce a triangle. The triangle free detour distance $D_{\Delta f}(u - v)$ is the length of a longest $u - v$ triangle free path in G. A $u - v$ path of length $D_{\Delta f}(u, v)$ is called a $u - v$ triangle free detour. A set $S \subseteq V$ of G is called a triangle free detour set of G if every vertex of G lies on a $u - v$ triangle free detour joining a pair of vertices of S. The triangle free detour number $dn_{\Delta f}(G)$ of $G$is the minimum order of its triangle free detour sets. This triangle free detour number was studied by S. Athisayanathan and S. Sethu Ramalingam in [8]. This concept was extended to square free detour number by K. Christy Rani and G. Priscilla Pacifica [4]. A square free detour number of G denoted by $dn_{\square f}(G)$ is defined as the minimum order of square free detour set S consisting of every pair of vertices of all the square free detours in which every vertex of G lies on.

In this article, we introduce the outer independent square free detour number denoted by $dn^{oi}_{\square f}(G)$. The outer independent square free detour number of some standard graphs and cycle related graphs are determined. For the basic terminologies we refer to Chartrand [2].

2. Preliminaries

The following theorems are used in the sequel.

Theorem 2.1 [3] For any connected graph $G, 2 \leq dn(G) \leq n$.

Theorem 2.2 [3] Every end-vertex of a non-trivial connected graph G belongs to every detour set of G.

Theorem 2.3 [3] If T is a tree with k end-vertices, then $dn(T) = k$.

Theorem 2.4 [4] If G is the cycle $C_n(n \geq 3)$, then $dn_{\square f}(G) = 2$.

3. Outer independent square free detour number of a graph

Definition 3.1 Let $G = (V, E)$ be a simple connected graph of order $n \geq 2$. A set of vertices S^* is called an outer independent square free detour set in G if S^* is a square free detour set such that either $V = S^*$ or $V - S^*$ is independent. The minimum cardinality of an outer independent square free detour number of G is called outer connected square free detour number of G and is denoted by $dn_{\Box f}^{oi}(G)$.

Example 3.2 For the graph G shown in Figure 1, the set $S_1^* = \{a, d, e, f, g, h, k\}$ is a minimum outer independent square free detour set and $S = \{a, d, e, g, h, k\}$ is a minimum square free detour set for G and so $dn_{\Box f}^{oi}(G) = 7$ and $dn_{\Box f}(G) = 6$. Here we find that $dn_{\Box f}(G) < dn_{\Box f}^{oi}(G)$. Moreover, the sets $S_2^* = \{a, d, e, f, h, i, k\}$, $S_3^* = \{a, d, e, f, i, j, k\}$ and $S_4^* = \{a, b, c, e, f, g, k\}$ are also the minimum outer independent square free detour sets of G. Hence there can be more than one minimum outer independent square free detour set for a graph G.

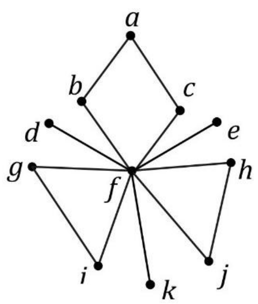

Figure 1: G

Theorem 3.3 *For any connected graph G, every end-vertex of G belongs to every outer independent detour set of G.*

Proof. Since every outer independent square free detour set is also a detour set of G, the proof follows from Theorem 2.2.

Theorem 3.4 *For any connected graph G, $2 \leq dn(G) \leq dn_{\Box f}^{oi}(G) \leq n$.*

Proof. The result follows from Theorems 2.1 and 3.3.

Remark 3.5 The bounds in Theorem 3.4 are sharp. The set of two end-vertices of a path P_2 is its minimum outer independent square free detour set so that $dn\,(P_2) = dn_{\Box f}^{oi}(P_2) = 2 = n$. The bounds in Theorem 3.4 are also strict. For the graph G of order 11 given in Figure 1, $2 < dn(G) < dn_{\Box f}^{oi}(G) < n$.

Theorem 3.6 *If G is a Path P_n, then* $dn^{oi}_{\square f}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Proof. Let $G = P_n$ be a path of order n and S^* be a set of $\left\lfloor \frac{n}{2} \right\rfloor + 1$ vertices. Then we consider two cases.

Case 1. Let n be odd. Let $S^* = \left\{x_{2i+1}: 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}$ be the square free detour set such that $V - S^*$ is independent. Hence S^* is an outer independent square free detour set. Thus $dn^{oi}_{\square f}(G) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Case 2. Let n be even. Let $S^* = \left\{x_1, x_{2j} : 1 \leq j \leq \frac{n}{2} \right\}$ be a square free detour set such that $V - S^*$ is independent. Thus $dn^{oi}_{\square f}(G) = \frac{n}{2} + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

The following corollary is immediate.

Corollary 3.7 For any connected graph G, $dn^{oi}_{\square f}(G) = n$ if and only if $G = P_2$.

Theorem 3.8 *If G is a star S_n, then* $dn^{oi}_{\square f}(G) = dn_{\square f}(G)$.

Proof. Let $G = S_n$ be a star with $n - 1$ end-vertices. Then by Theorem 3.3, $S^* = \{x_i: 1 \leq i \leq n - 1\}$ where $x_i, x_2, \ldots, x_{n-1}$ are the end-vertices of S_n such that $V - S^*$ is independent. Hence $dn^{oi}_{\square f}(G) = n - 1$. By Theorem 2.3, S^* is also a minimum detour set and so $dn_{\square f}(G) = n - 1$. Hence $dn^{oi}_{\square f}(G) = dn_{\square f}(G)$.

Theorem 3.9 If G is a complete bipartite graph $\mathrm{K_{m,n}} (2 \leq \mathrm{m} \leq \mathrm{n})$, then $\mathrm{dn^{oi}_{\square f}(G) = m}$.

Proof. Let $G = K_{m,n}$ be a complete bipartite graph of order with two partitions X and Y where $|X| = m$ and $|Y| = n$. Let S^* be a set of m vertices of X. Now, it is easy to verify that $V - S^*$ is independent. Hence $dn^{oi}_{\square f}(G) = m$.

Remark 3.10 Due to the connectivity of the complete graph K_n, it is not possible to find the outer independent square free detour number for K_n.

Theorem 3.11 *If G is a cycle C_n, then* $dn^{oi}_{\square f}(G) = \left\lceil \frac{n}{dn_{\square f}(G)} \right\rceil$.

Proof. Let $G = C_n$ be a cycle of order n. Let S^* be any set of $\left\lceil \frac{n}{2} \right\rceil$ vertices of G. we consider two cases.

Case 1. Let n be odd. Let $S^* = \left\{x_{2i+1}: 0 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \right\}$ be a square free detour set such that $V - S^*$ is independent. Hence $dn^{oi}_{\square f}(G) = \left\lceil \frac{n}{2} \right\rceil$.

Case 2. Let n be even. Let $S^* = \left\{x_{2j}: 1 \leq j \leq \frac{n}{2} \right\}$ be a square free detour set such that $V - S^*$ is independent. Hence $dn^{oi}_{\square f}(G) = \frac{n}{2}$.

From the above cases, we observe that $dn^{oi}_{\square f}(G) = \left\lceil \frac{n}{2} \right\rceil$. Then by Theorem 2.4, it follows that $dn^{oi}_{\square f}(G) = \left\lceil \frac{n}{dn_{\square f}(G)} \right\rceil$.

Theorem 3.12 *If G is a Wheel, then* $dn^{oi}_{\square f}(G) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. Let $G = W_n$ be a Wheel of order n. Then $S^* = \left\{x_0, x_{2i+1}: 0 \le i \le \left\lfloor \frac{n}{2} \right\rfloor\right\}$ is a square free detour set such that $V - S^*$ is independent. Thus $dn^{oi}_{\square f}(G) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 3.13 *If G is a Flower graph Fl_n, then* $dn^{oi}_{\square f}(G) = \begin{cases} n+1 & if\ n\ is\ even \\ n+2 & if\ n\ is\ odd \end{cases}$

Proof. Let $G = Fl_n$ be a Flower graph of order n. Let x_0 be the hub, $x_i(1 \le j \le n)$ be the vertices on the inner rim and $y_j(1 \le j \le n)$ be the vertices at square free detour distance 3 from the hub. Then we have two cases.

Case 1. Let n be even. Let $S^* = \left\{x_0, x_{2i+1}, y_{2j}: 0 \le i \le \frac{n}{2}: 1 \le j \le \frac{n}{2}\right\}$ is the square free detour set such that $V - S^*$ is independent. Thus $dn^{oi}_{\square f}(G) = n + 1$.

Case 2. Let n be odd. Then $S^* = \left\{y_1, x_{2i+1}, y_{2j}: 0 \le i \le \left\lfloor \frac{n}{2} \right\rfloor, 1 \le j \le \frac{n}{2}\right\}$ is a square free detour set such that $V - S^*$ is independent. Hence $dn^{oi}_{\square f}(G) = n + 2$.

Theorem 3.14 *If G is a Helm H_n, then* $dn^{oi}_{\square f}(G) = \begin{cases} \frac{3(n+1)}{2} & if\ n\ is\ odd \\ \frac{3n}{2} + 1 & if\ n\ is\ even \end{cases}$

Proof. Let $G = H_n$ be a Helm of order n. Let S^* be any set in G. Then we have the following two cases.

Case 1. Let n be odd. Then $S^* = \left\{x_0, x_{2i+1}, y_j: 0 \le i \le \left\lceil \frac{n}{2} \right\rceil, 1 \le j \le n\right\}$ is a square free detour set of $\frac{3(n+1)}{2}$ vertices where x_0 is the hub, $x_i(1 \le j \le n)$ are the vertices on the rim and $y_j(1 \le j \le n)$ are the pendent vertices of H_n such that $V - S^*$ is independent. Hence $dn^{oi}_{\square f}(G) = \frac{3(n+1)}{2}$.

Case 2. Let n be even. Then $S^* = \left\{x_0, x_{2i+1}, y_j: 0 \le i \le \frac{n}{2}, 1 \le j \le n\right\}$ is a square free detour set of $\frac{3n+2}{2}$ vertices where x_0 is the hub, $x_i(1 \le j \le n)$ are the vertices on the rim and $y_j(1 \le j \le n)$ are the pendent vertices of H_n such that $V - S^*$ is independent. Hence $dn^{oi}_{\square f}(G) = \frac{3n}{2} + 1$.

Theorem 3.15 *If G is a Closed Helm CH_n, then* $dn^{oi}_{\square f}(G) = \begin{cases} n+1 & if\ n\ is\ even \\ n+2 & if\ n\ is\ odd \end{cases}$

Proof. Let $G = CH_n$ be a closed Helm of order n. Let x_0 be the hub, $x_i(1 \leq j \leq n)$ and $y_j(1 \leq j \leq n)$ are the vertices of inner and outer rim of CH_n respectively. Then we consider two cases.

Case 1. Let n be even. Let $S^* = \left\{x_0, x_{2i+1}, y_{2j}: 0 \leq i \leq \frac{n}{2}: 1 \leq j \leq \frac{n}{2}\right\}$ is the square free detour set such that $V - S^*$ is independent. Thus $dn^{oi}_{\square f}(G) = n + 1$.

Case 2. Let n be odd. Then $S^* = \left\{x_0, y_1, x_{2i+1}, y_{2j}, y_1: 0 \leq i \leq \left\lfloor\frac{n}{2}\right\rfloor: 1 \leq j \leq \frac{n-1}{2}\right\}$ is a square free detour set such that $V - S^*$ is independent. Hence $dn^{oi}_{\square f}(G) = n + 2$.

Theorem 3.16 *For any pair of integers α and β with $2 \leq \alpha \leq \beta$, there exists a connected graph G of order $\beta + 3$ with square free detour number α and outer independent square free detour number β.*

Proof. We consider two cases.

Case 1. $2 \leq \alpha = \beta$. Any star S_n with $n - 1$ end-vertices has the desired property.

Case 2. $2 \leq \alpha < \beta$. Let H_1 be a graph obtained from $C_6: x_1, x_2, x_3, x_4, x_5, x_6, x_1$ by adding $\alpha - 1$ new vertices $y_1, y_2, \dots, y_{\alpha-1}$ to x_6. Let H_2 be the graph derived from H_1 by adding $\beta - \alpha - 1$ new vertices $z_1, z_2, \dots, z_{\beta-\alpha-1}$ and identifying z_1 with x_3. Let G be the graph derived from H_2 by joining the remaining $\beta - \alpha - 2$ vertices $z_2, z_3, \dots, z_{\beta-\alpha-1}$ to x_2 and x_4 of C_6. The resulting graph G of order $\beta + 3$ is shown in Figure 2. By Theorem 3.3, it is verified that $S_1 = \{x_1, y_1, y_2, \dots, y_{\alpha-1}\}$ and $S_2 = \{x_5, y_1, y_2, \dots, y_{\alpha-1}\}$ are the square free detour sets of G and so $dn_{\square f}^{oi}(G) = \alpha$.

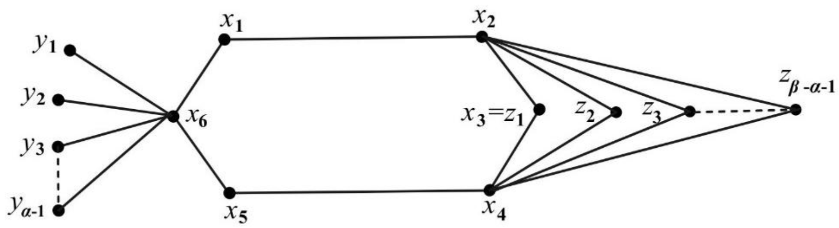

Figure 2: G

Now, consider $S' = \{z_1, z_2, \dots, z_{\beta-\alpha-1}\}$, the set of all vertices finally added to C_6 to obtain G. It is easy to verify that $S^* = S' \cup (S_1 \cup S_2)$ is the outer independent square free detour set of G and so $dn^{oi}_{\square f}(G) = \beta$.

4. Conclusion

In this paper, we determined the outer independent square free detour number of some standard graphs and cycle related graphs. The relationship between the square free detour number and the outer independent square free detour number has been established. Further investigation is open for any other class of graphs.

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