The Outer Connected Detour Monophonic Number of a Graph

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Abstract

For a connected graph $G = (V, E)$ of order $n \geq 2$, a set $M \subseteq V$ is called a *monophonic set* of Gif every vertex of $G$is contained in a monophonic path joining some pair of vertices in M. The monophonic number $m(G)$ of G is the minimum cardinality of its monophonic sets. If $M = V$ or the subgraph $G[V - M]$is connected, then a detour monophonic set M of a connected graph G is said to be an *outer connected detour monophonic set*of G.The *outer connecteddetourmonophonic number* of G, indicated by the symbol $ocd_m(G)$, is the minimum cardinality of an outer connected detour monophonic set of G. The outer connected detour monophonic number of some standard graphs are determined. It is shown that for positive integers r_m, $d_m$and $l \geq 2$ with $r_m < d_m \leq 2r_m$,there exists a connected graph $G$with$rad_mG = r_m$, $diam_mG = d_m$and $ocd_m(G)= l$. Also, it is shown that for every pair of integers $a$and b with $2 \leq a \leq b$, there exists a connected graph $G$with$dm(G) = a$ and$ocd_m(G) = b$.

Keywords: chord, monophonic path, monophonic number, detour monophonic path, detour monophonic number, outer connected detour monophonic number.

AMS Subject Classification: 05C38[3]

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[3] Received on June15th, 2022. Accepted on Sep 1st, 2022. Published on Nov 30th, 2022. doi: 10.23755/rm.v44i0.921. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

A finite, undirected connected graph with no loops or many edges is referred to as a graph $G = (V, E)$. By n and m, respectively, we indicate the order and size of G. We refer to [1] for the fundamental terms used in graph theory. If uv is an edge of G, then two vertices u and v are said to be adjacent. If two edges of G share a vertex, they are said to to be adjacent. Let $S \subset V$ be any subset of vertices of G. Then the graph with S as its vertex set and all of its edges in E having both of their end points in S is the *induced subgraph* $G[S]$. A vertex v is an extreme vertex of a graph G if the subgraph induced by its neighbors is complete.

The length of the shortest path in a connected graph G is equal to the distance $\mathrm{d(u, v)}$ between two vertices u and v. An $\mathrm{u - v}$geodesic is a $\mathrm{u - v}$ path with length $\mathrm{d(u, v)}$. An edge that connects two non-adjacent vertices of a path P is called the chord of P . A chordless$\mathrm{u - v}$ path is referred to as a *monophonic path*. The *monophonic distance* $d_m(u, v)$for two vertices u and v in a connected graph G is the length of a longest $u - v$ monophonic path in G. An $\mathrm{u - v}$detour monophonic path is one that has a length of $\mathrm{d_m(u, v)}$.The monophonic eccentricity of a vertex v, denoted by $\mathrm{e_m(v)}$ is the monophonic distance between v and a vertex farthest from v. The monophonic radius, $rad_m(G)$, and the monophonic diameter, $diam_m(G)$ are the vertices respective minimum and maximum monophonic eccentricities.

The closed interval $J_{dm}[u, v]$ for two vertices u and v is consists of all the vertices along an $u - v$ detour monophonic path, including the vertices u and v. If $v \in E$, then$J_{dm}[u, v] = \{u, v\}$. For a set M of vertices, let $J_{dm}[M] = \cup_{u,v \in M} J[u, v]$. Then certainly $M \subseteq J_{dm}[M]$. If $J_{dm}[M] = V$,a set $M \subseteq V(G)$ is referred to as a *detour monophonic set* of G.The *detour monophonic* number $dm(G)$ of G is the minimum order of its detour monophonic sets.Any detour monophonic set of order $d_m(G)$ is referred to as an d_m-set of G. In [2-4], these concepts were investigated. The following theorem is used in sequel.

Theorem 1.1. [4] Each extreme vertex of a connected graph G belongs to everydetour monophonic set of G.

2. The Outer Connected Detour Monophonic Number of a Graph

Definition 2.1. If $M = V$ or the subgraph $G[V - M]$is connected, then a detour monophonic set M of a connected graph G is said to be an *outer connected detour monophonic set*of G.The *outer connecteddetourmonophonic number* of G, indicated by the symbol $ocd_m(G)$, is the minimum cardinality of an outer connected detour monophonic set of G. The ocd_m*-set* of G is a minimum cardinality of an outer connected detour monophonic setof G.

Example 2.2. $M = \{v_2, v_4\}$ is a dm-set of the graph G in Figure 2.1 such that $dm(G) = 2$. M is not an outer connected detour monophonic set of G because $G[V - M]$ is not connected, and as a result, $ocd_m(G) \geq 3$. Now since $M_1 = \{v_2, v_3, v_4\}$, is a ocd_m-set of G, and
$ocd_m(G) = 3$ as a result.

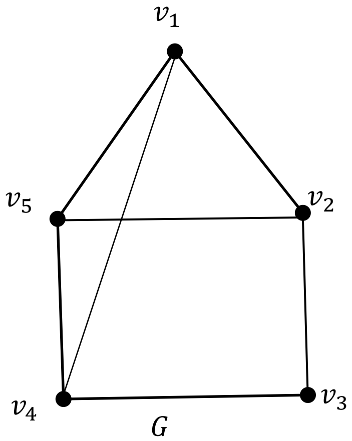

Figure 2.1

Observation 2.3.
(i) Each extreme vertex of a connected graph G belongs to every outer connected detour monophonic set of G.
(ii) No cut vertex of G belongs to any ocd_m-set of G.
(iii)For any connected graph G of order$n$, $2 \leq dm(G) \leq ocd_m(G) \leq n$.

Theorem 2.4. $ocd_m(G) = n$, for the complete graph $G = K_n$ $(n \geq 2)$.
Proof. The vertex set of $K_n$is the unique outer connected detour monophonic set of K_n since every vertex of the complete graph $K_n(n \geq 2)$is the extreme vertex. Therefore, $ocd_m(G) = n$. ■

Theorem 2.5. $ocd_m(T) = k$, for any tree T with k end vertices.
Proof. Let M represent the collection of $T's$ end vertices. According to Observation 2.3(i) and (ii), $ocd_m(T) \geq |M|$. M is the unique outer connected detour monophonic set of T,since the subgraph $G[V - M]$ is connected. Consequently, $ocd_m(T) = |M| = k$.■

Corollary2.6. $ocd_m(P_n) = 2$ for the non-trivial path P_n $(n \geq 3)$.

Corollary2.7. $ocd_m(K_{1,n-1}) = n - 1$ for star $K_{1,n-1}(n \geq 3)$.

Theorem 2.8. $ocd_m(G) = 3$, for the cycle $G = C_n(n \geq 4)$.

Proof. Set the cycle C_n to be $v_1, v_2, \ldots, v_n, v_1$. Then, $M = \{v_1, v_2, v_3\}$ is a $G's$ outer connected detour monophonic set, resulting in $ocd_m(G) \leq 3$. We establish$ocd_m(G) = 3$. Assume that $ocd_m(G)$= 2. Then $G'socd_m$-set is $M_1 = \{x, y\}$. It is obvious that x and y are not adjacent. A contradiction results since $G[V - M_1]$is not connected and M_1 is not a $G's\ ocd_m -$ set. ,Consequently, $ocd_m(G) = 3$. ■

Theorem 2.9. $ocd_m(G) = \begin{cases} s, & if\ r = 1, s \geq 2 \\ 3, & if\ r = s = 2 \\ 4, & if\ 2 < r \leq s \end{cases}$for the complete bipartite graph G= $K_{r,s}$

Proof. $G = K_{r,s}$is a tree with s end vertices when $r = 1$ and $s \geq 2$.Therefore,$ocd_m(K_{1,s}) = s$ as per Corollary2.7.$G = K_{2,2}$is the cycleC_4 when $r = s = 2$, Thus, according to Theorem 2.8, $ocd_m(K_{2,2}) = 3$. Let $2 < r \leq s$.Let $X = \{x_1, x_2, \ldots, x_m\}$,$Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartitions of G. Let $M = \{x_i, x_j, y_k, y_l\}$, where$i \neq j, k \neq l$.Then M is adetour monophonic set of G. M is anouter connected detour monophonic set of G because the subgraph$G[V - M]$is connected, and as a result, $ocd_m(G) \leq 4$. We demonstrate that $ocd_m(G) = 4$. Let's assume that $ocd_m(G) \leq 3$.Then $|M| \leq 3$ and there exists a$ocd_m(G)$-set M. If $M \subseteq X$ or $M \subseteq Y$ then $G[V - M]$ is not connected. Consequently, $M \subset XUY$.Which suggests M is not aouter connected detour monophonic set of G, which is in contrast with the statement made earlier. Thus$ocd_m(G) = 4$. ■

Theorem 2.10 $ocd_m(G) = \begin{cases} 2\ if\ n = 5 \\ 3\ if\ n \geq 6 \end{cases}$for the wheel $G = K_1 + C_{n-1}$ $(n \geq 5)$.

Proof. Let's say that $V(K_1) = x$ and $V(C_{n-1}) = \{v_1, v_{2,} \ldots \ldots \ldots, v_{n-1}\}$. $M = \{v_1, v_{3,}\}$ is an $ocd_m - set$ of G for $n = 5$. Therefore, $ocd_m(G) = 2$. So, let $n \geq 6$. Hence it follows that $ocd_m(G) \geq 3$ Let $M_1 = \{v_1, v_{2,} v_3\}$. Then M_1 is an outer connected detour monophonic set of G. Consequently, $ocd_m(G) = 3$. ■

Theorem 2.11 $ocd_m(G) = \begin{cases} 2\ if\ n = 4 \\ 3\ if\ n > 4 \end{cases}$,for the graph $G = K_1 + P_{n-1}$.

Proof. Let's say that $V(K_1) = x$ and $V(P_{n-1}) = \{v_1, v_{2,} \ldots \ldots \ldots, v_{n-1}\}$.$M_1 = \{v_1, v_3\}$ is a$ocd_m - set\ of\ G$ for n = 4 , and $ocd_m(G) = 2$. So, let $n \geq 5$. Let $M = \{v_1, v_{n-1}\}$ be the extreme vertices of G.By observation 2.3(i)M is a subset of every $ocd_m - set\ of\ G$. Since M is not a outer connected detour monophonic set of G, $ocd_m(G) \geq 3$. Now $M_2 = M \cup \{x\}$ is a $ocd_m - set\ of\ G. so\ that\ ocd_m(G) = 3$. ■

Theorem 2.12. Consider the connected graph G, where $dm(G) = 2$. If deg $(x) \geq 3$ for every $x \in V$, then $ocd_m(G) = 2$.

Proof. Let the detour monophonic set of G be $M\{u, v\}$,$deg(x) \geq 3$ for$x \in V$, so $G[V - M]$is connected. As a result, M is an outer connected detour monophonic set of G so that

$ocd_m(G) = 2$. ■

Theorem 2.13. Suppose G is a connected graph with $dm(G) = 2$. If G has 2 possible outermost vertices $u, v \in V$ that are not relatedand $\Delta[\langle V - \{u, v\}\rangle] = n - 3$.Then$ocd_m(G)$= 2.
Proof. Let u and v represent the outermost vertices of G. Let $M = \{u, v\}$. M is therefore a G detour monophonic set. Given that $\Delta\,[\langle V - \{u, v\}\rangle] = n - 3$, and that u and v are not adjacent outermost vertices of G, $G[V - M]$is connected. As a result, M is a outer connected detour monophonic set of G, and$ocd_m(G) = 2$. ■

Theorem 2.14. Let G be a connected graph of order n that has precisely one vertex that is not a cut vertex and has a degree of n - 1. Then$ocd_m(G) \leq n - 3$.
Proof. Let x represent the non-cut vertex of G at the vertex of degreen– 1.Since $N(x)$is not complete, there exist at least two non-adjacent vertices, say y and z that are both members of $N(x)$ .There are at least two vertices say $x_1$and x_2 because $N(x)$is the unique vertex of degree $n - 1$, and they are both located on the $y - z$detour monophonic path such that $x_1 \neq x$, $x_2 \neq x$. $M = V(G) - \{x, x_1, x_2\}$ is a detour monophonic set of G. M is an outer connected detour monophonic set of G since the subgraph $G[V–M]$is connected, which causes$ocd_m(G) \leq n - 3$. ■

Theorem 2.15. Let G be an order $n \geq 3$. $ocd_m(G) \leq n - 1$ if G contains a cut vertex of degree n – 1.
Proof. Let M be a minimum outer connected detour monophonic set and v be the cut vertex of degree $n - 1$ in G. Observation 2.3(ii) says that $v \notin M$. It is obvious that$ocd_m(G) \leq n - 1$, $M = V(G) - \{v\}$ is a detour monophonic set of G. M is an outer connected detour monophonic set of G Since v is a universal cut vertex of G, the subgraph $G\langle V - M\rangle$is connected. As a result, M is an outer connected detour monophonic set of G. which causes $ocd_m(G) \leq n - 1$. ■

Theorem 2.16 There exists a connected graph G with $rad_m G = r_m$, $dia_m G = d_m$and $ocd_m(G) = l$ for positive integers $r_m, d_m$and $l \geq 2$ with $r_m < d_m \leq 2r_m$.
Proof. We make the convenient assumptions that $r_m = r$ and $d_m = d$. Let $G = K_{1,l}$ when $r = 1$. Theorem 2.5 states that $ocd_m(G) = l$. Let $r_m \geq 2$. Let $C_{r+2}: v_1, v_2, \ldots v_{r+2}$be a cycle of length $r + 2$ and let $P_{d_m - r_m + 1}: u_0, u_1, u_2, \ldots, u_{d_m - r_m}$be that cycle. By locating v_1 in $C_{r+2}$and $u_0$in $p_{d_{m-r_m+1}}$, we may construct the graph H.The graph shown in Figure 2.2 is then created by joining each of the $w_i$vertices ($1 \leq i \leq l - 3$) to the vertex $u_{d_{m-r_{m-1}}}$ and adding new vertices$w_1, w_2, \ldots, w_{l-3}$to $u_{d_{m-r_{m-1}}}$So, $rad_m G = r_m$, $dia_m G = d_m$. The set of all $G's$ end vertices, $W = \{w_1, w_2, \ldots, w_{l-3}, u_{d_{m-r_m}}\}$shall be defined. W is then contained in every detour monophonic detour set of G according to Observation 2.3(i). Since$J_{dm}[M] \neq V, W$ is not an outer connected detour monophonic set of G and so $ocd_m(G) \geq l - 1$. $ocd_m(G) \geq l$ because it is obvious that M is not an outer connected detour

monophonic set of G, where$M = W \cup\{u_0\}$ and$u_0 \notin M$. $ocd_m(G) = l$ because it is obvious that M is an outer connected detour monophonic set of G, where $M = W \cup\{v_2, v_3\}$.

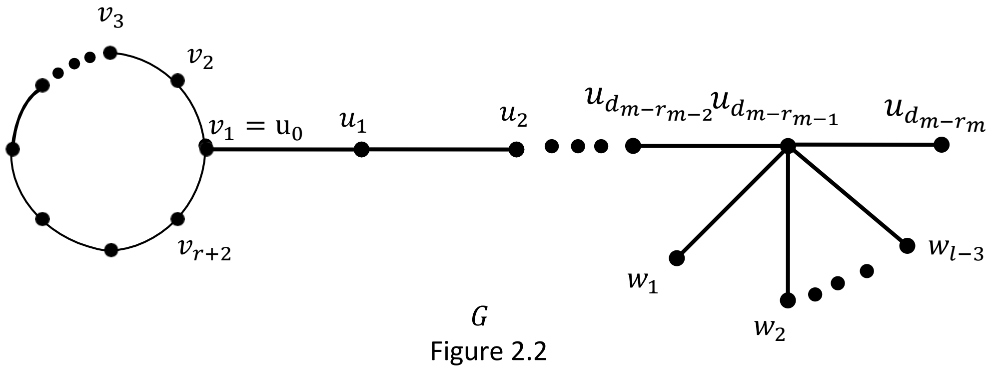

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Figure 2.2

Theorem 2.17. There is a connected graph G with $dm(G) = a$ and $ocd_m(G) = b$ for every pair of positive integers a and b such that $2 \le a \le b$.

Proof. Let $V(\overline{K_2}) = \{x, y\}$. Let a graph be created by adding additional vertices to $(\overline{K_2})$ as follows: $z_1, z_2, \dots, z_{a-1}, v_1, v_2, \dots, v_{b-a}$, and connecting each z_i $(1 \le i \le a-1)$ with x and y. Graph G is displayed in Figure 2.3. First, we demonstrate that $dm(G) = a$. Assume that $Z = \{z_1, z_2, \dots, z_{a-1}\}$ is the collection of all end vertices of G. According to Theorem1.1, every detour monophonic set of G has Z as a subset. Since it is obvious that Z is not a monophonic detour set of $G, dm(G) \ge a.$ Now that $Z\cup\{y\}$ is a monophonic set, $dm(G) = a$.

We then demonstrate that $ocd_m(G) = b. Z \cup \{y\}$is not an outer connected detour monophonic set of G because $G[V - (Z \cup \{y\})]$ is not connected. According to Observation 2.3(i), each outer connected detour monophonic set of G has the vertex z_i $(1 \le i \le a-1)$. Additionally, it is simple to see that any outer connected detour monophonic set of G contains each $v_i (1 \le i \le b-a)$,which means that $ocd_m(G) \ge a - 1 + b - a = b - 1$. Let $M = Z \cup \{v_1, v_2, \dots, v_{b-a}\}$. Since M is not an outer connected detour monophonic set of G, then $ocd_m(G) \ge b$. Now that $M \cup \{x\}$ is an outer connected detour monophonic set of G $ocd_m(G) = b$.

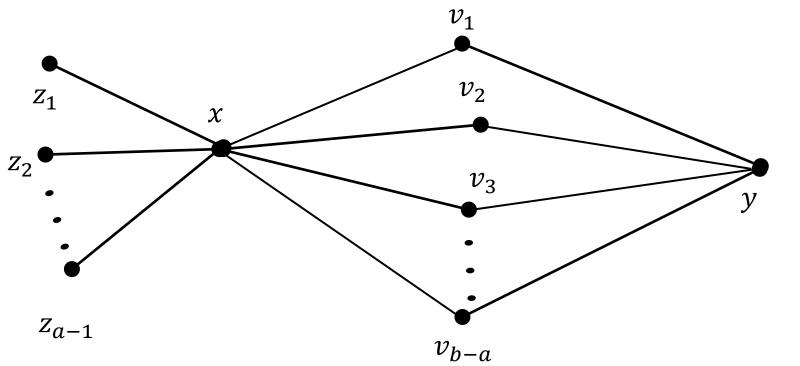

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Figure 2.3

3. Conclusion

This article established a novel detour monophonic distance parameter called the outer connected detour monophonic number of graphs. We will develop this concept to incorporate more distance considerations in a subsequent investigation.

Acknowledgements

We are thankful to the referees for their constructive and detailed comments and suggestions which improved the paper overall.

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