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Semi generalization of δI*-closed sets in ideal topological space

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Abstract

In this paper we introduce the notion of semi generalized δI*-closed sets or gsδI*-closed sets using semi open sets and investigate its basic properties and characterizations in an ideal topological space. This class of sets is properly lies between the class of δI*-closed sets and the class of g-closed sets. Also, study the relationship with various existing closed sets in ideal topological spaces. Moreover, we introduce and study the concept of maximal gsδI*-closed sets.

Keywords: ideal topological space, δI*-closed sets, gsδI*-closed sets.

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1. Introduction and Preliminaries

An ideal I is a non-empty collection of subsets of X which satisfies: (i) A∈I and B ⊆ A implies B ∈I, and (ii) A∈I and B ∈I implies A ∪ B ∈I. Given a topological space (X, τ) with an ideal I on X called ideal topological space denoted by (X, τ, I). Kuratowski [5] and vaidhyanathaswamy [18] was studied the notion of ideal topological spaces, J. Dontchev, M. Ganster [3], Navaneethakrishnan, P. Paulraj Joseph [13], D. Jankovic, T. R. Hamlett [4], M. N. Mukherjee, R. Bishwambhar, R. Sen [10], A. A. Nasef, R. A. Mahmond [12] etc., were investigated applications to various fields of ideal topology. If P(X) is the collection of all subsets of X a set operator (.)*: P(X) → P(X) called a local function [5] for any subset A of X with respect to I and τ is defined as, A*(I, τ) = {x ∈X: U ∩ A ∉ I for every U ∈ τ(x)}, where τ(x) = {U ∈ τ / x ∈ U}. A kuratowski closure operator cl*(A) for a topology τ*(I, τ) called *-topology finer than τ is defined by cl*(A) = A ∪A*(I, τ). A subset A of X is said to be δ-closed [19] set if $cl_\delta(A)$ = A, where $cl_\delta(A)$ = {x ∈X: Int(cl(U)) ∩ A ≠ϕ, for every U ∈τ(x)}. The complement of δ-closed set is δ-open set. A subset A subset A of a space (X, τ) is an α-open [14] (resp. semi open [7]) set if A ⊂int(cl(int(A))) (resp. A ⊂ cl(int(A))). The complement of a semi open (resp.α-open) set is called a semi closed (resp.α-open).

Definition 1.1. Let (X,τ) be a topological space. A subset A of X is said to be
(i) a generalized closed (briefly, g-closed) set [6] if cl(A) ⊂ U whenever A ⊂ U and U is open in (X, τ).
(ii) a generalized semi closed (briefly, gs-closed) set [1] if scl(A) ⊂ U whenever A ⊂ U and U is open set in (X, τ).
(iii) a semi-generalized closed (briefly, sg-closed) set [2] if scl(A) ⊂ U whenever A ⊂ U and U is semi open set in (X, τ).
(iv)anα-generalized closed (briefly,αg-closed) se [8]t if αcl(A) ⊂ U whenever A ⊂ U and U is open in (X, τ).
(v) a generalized α-closed (briefly, gα-closed) set [9] if αcl(A) ⊂ U whenever A ⊂ U and U is α-open in (X, τ).
(vi) a $\hat{g}$ (or) w-closed set [20] if cl(A) ⊂ U whenever A ⊂ U and U is semi open set in (X, τ).

Definition 1.2. [21] Let (X, τ, I) be an ideal topological space. A subset A of X is said to be an I_g-closed set if A* ⊂ U whenever A ⊂ U and U is open in X.

Definition 1.3. [21] Let (X, τ, I) be an ideal topological space, A a subset of X and x is a point of X. Then

(1) x is called a δ-I-cluster point of A if A ∩int(cl*(U)) ≠ ϕ, for each open neighborhood U of x.
(2) the family of all δ-I-cluster points of A is called the δ-I-closure of A and is denoted by $[A]_{\delta\text{-}I}$.
(3) a subset A is said to be δ-I-closed if $[A]_{\delta\text{-}I}$ = A. The complement of a δ-I-closed set of X is said to be δ-I-open.

Lemma 1.4. [21] Let A and B be subsets of an ideal topological space (X, τ, I). Then, the following properties hold.
(1) A ⊂ $[A]_{\delta\text{-}I}$.
(2) If A ⊂ B, then $[A]_{\delta\text{-}I}$⊂ $[B]_{\delta\text{-}I}$.
(3) $[A]_{\delta\text{-}I}$ = ∩ {F ⊂ X / A ⊂ F and F is δ-I-closed}.
(4) If A_α is δ-I-closed set of Xs for each α∈Δ, then ∩ {A_α / α∈Δ} is δ-I-closed.
(5) $[A]_{\delta\text{-}I}$ is δ-I-closed.

Lemma 1.5. [21] Let (X, τ, I) be an ideal topological space and $\tau_{\delta\text{-}I}$ = {A ⊂ X / A is δ-I-open subset of (X, τ, I)}. Then $\tau_{\delta\text{-}I}$ is a topology such that $\tau_S \subset \tau_{\delta\text{-}I} \subset \tau$, where τ_S is the collection of
δ-open sets.

Definition 1.6. [16] Let (X, τ, I) be an ideal topological space and A a subset of X. Then [A]*(I, τ) = {x ∈X: int$[U]_{\delta\text{-}I}$∩ A ≠ϕ for every U ∈τ(x)} is called local δI-closure function of A with respect to the ideal I and topology τ, where τ(x) = {U ∈ τ / x ∈ U}. A subset A is said to be δI-closed if [A]* = A. The complement of δI-closed set is called δI-open set.

Remark 1.7.[16] Always, (i) [A]* is closed, (ii) [ϕ]* = ϕ and [X]* = X, (iii) A ⊆ [A]*.

Lemma.1.8. [16] Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then for local δI-closure functions the following properties hold.
(i) If A ⊆B then [A]*⊆[B]*.
(ii) [A ∪B] *= [A]* ∪ [B]*.
(iii) [A ∩ B] * ⊆[A]* ∩ [B]*.
(iv) [[A]*] * = [A]*.

Lemma 1.9.[16] (i) cl(A) ⊆ [A]*,
(ii) A* ⊆ [A]*,
(iii) cl_δ(A) ⊆ [A]*,
(iv) $[A]_{\delta\text{-}I}$⊆ [A]*.

Definition 1.10. [17] A subset A of an ideal space (X, τ, I) is called gδI*-closed if [A]* ⊆ U whenever A ⊆ U and U is open in (X, τ, I). The complement of a gδI*-closed set in (X, τ, I) is called gδI*open set in (X, τ, I).

2. gsδI*- closed Sets

In this section we introduce gsδI*-closed sets and discuss the relationship with some existing sets.

Definition 2.1. A subset A of an ideal topological space (X, τ, I) is called gsδI*-closed if [A]*⊆ U whenever A ⊆ U and U is semi open set in (X, τ, I). The complement of gsδI*-closed set in (X, τ, I), is called gsδI*-open set in (X, τ, I).

Theorem 2.2. EveryδI*-closed set is gsδI*-closed.
Proof. Let A be any δI*-closed set and U be any semi open set containing A. Since A is δI*-closed, [A]* = A. Therefore, A is gsδI*-closed set in (X, τ, I).

Remark 2.3. The converse of the above Theorem 2.2 is need not be true as shown in the following Example 2.4.

Example 2.4. Let X = {a, b, c}, τ = {X, ϕ, {b}, {c, d}, {b, c}, {b, c, d}}, I = {ϕ, {d}}. Let A = {a, b, c}. Then, A is gsδI*-closed but not δI*-closed.

Theorem 2.5. In an ideal topological space (X, τ, I), every gsδI*-closed set is
(i) ĝ -closed set in (X, τ).
(ii) g-closed (resp. gα, αg, sg, gs) -closed set in (X, τ).
(iii) I_g -closed set in (X, τ, I).
Proof. (i) Let A be a gsδI*-closed set and U be any semi open set in (X, τ, I) containing A. Since A is gsδI*-closed, [A]*⊆ U. Then cl(A) ⊆ U and hence A is ĝ -closed in (X, τ, I), by Lemma 1.9.
(ii) By [20], every ĝ-closed set is g-closed (resp. gα-closed, αg-closed, sg-closed, gs-closed) set in (X, τ, I). Therefore, it holds.
(iii) Since every g-closed set is I_g-closed, it holds.

Remark 2.6. The following Example 2.7 shows that, the converse of the above Theorem 2.5 (i) is not always true.
Example 2.7. Let X = {a, b, c, d}, τ = {X, ϕ, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}} and I = {ϕ, {b}}. Let A = {c, d}. Then A is ĝ-closed set but not gsδI*-closed.

Remark 2.8. The following Examples shows that, the converse of Theorem 2.5 (ii) is not true.

Example 2.9. Let X = {a, b, c, d}, τ = {X, ϕ, {b}, {c}, {b, c,}} and I = {ϕ, {d}}. Let A = {d}. Then A is g-closed, αg-closed, gα-closed but not gsδI*-closed.

Example 2.10. Let X = {a, b, c, d}, τ = {X, ϕ, {a}, {c, d}, {a, c, d}, {b, c, d}} and I = {ϕ, {a}}. Let A = {a, b}. Then A is gs-closed and sg-closed but not gsδI* closed.

Remark 2.11. The following Example 2.12 shows that, the converse of Theorem 2.5 (iii) is not always true.

Example 2.12. Let X = {a, b, c, d}, τ = {X, ϕ, {b}, {a, b}, {b, c}, {a, b, c}, {a, b, d}} and I = {ϕ, {b}}. Let A = {b}. Then A is I_g -closed but not gsδI*-closed.

3. Characterizations

In this section we study some of the basic properties and characterizations of gsδI*-closed sets.

Theorem 3.1. Let (X, τ, I) be an ideal space and A a subset of X. Then [A]* is semi closed.
Proof. By Remark 1.7, [A]* is closed and hence it is semi closed.

Theorem 3.2. Let (X, τ, I) be an ideal space and A ⊆ X. If A ⊆ B ⊆[A]*, then [A]* = [B]*.
Proof. Since A ⊆ B, [A]*⊆[B]* and since B ⊆[A]*, [B]*⊆[[A]*] * = [A]*, By Lemma 1.8 and Lemma 1.9. Therefore, [A]* = [B]*.

Theorem 3.3. Let (X, τ, I) be an ideal space. Then [A]* is always gsδI*-closed for every subset A of X.
Proof. Let [A]*⊆ U, where U is semi open. Always, [[A]*] * = [A]*. Hence [A]* is gsδI*-closed.

Theorem 3.4. Let (X, τ, I) be an ideal space and A ⊆ X. If sker(A) is gsδI*-closed, then A is also gsδI*-closed.
Proof. Suppose that, sker(A) is a gsδI*-closed set. If A ⊆ U and U is semi open, then sker(A) ⊆ U. Since sker(A) is gsδI*-closed, [sker(A)]*⊆ U. Always, [A]*⊆[sker(A)]*. Thus, A is gsδI*-closed.
The following Example 3.5 shows that, the converse of the above Theorem 3.4 is not always hold.

Example 3.5. In Example 2.12, let A = {a, b}. Then A is gsδI*-closed. But, sker(A) = {a ,b, c} is not gsδI*-closed.

Theorem 3.6. If A is gsδI*-closed subset in (X, τ, I), then [A]* – A does not contain any nonempty closed set in (X, τ, I).
Proof. Let F be any closed set in (X, τ, I) such that F ⊆[A]* – A then A ⊆ X – F and X – F is open and hence semiopen in (X, τ, I). Since A is gsδI*-closed, [A]*⊆X – F. Hence, F ⊆ X – [A]*. Therefore, F ⊆ ([A]* – A) ∩ (X – [A]*) = ϕ.

Remark 3.7. The converse of the above Theorem 3.6 is not always true as shown in the following Example 3.8.

Example 3.8. Let X = {a, b, c}, τ = {X,ϕ, {a}, {b}, {a, b}} and I = {ϕ, {c}, {d}, {c, d}}. Let A = {a, b, c}. Then [A]* – A = X – {a, b, c} = {d} does not contain any nonempty closed set. But A is not a gsδI*-closed subset of (X, τ, I).

Theorem 3.9. For a subset A of an ideal space (X, τ, I), cl(A) – A is gsδI*-closed if and only if A ∪ (X – cl(A)) is gsδI*-open.
Proof. Necessity - Let F = cl(A) – A. By hypothesis, F is gsδI*-closed and X – F = X ∩ (X – F) = X ∩ (X – (cl(A) – A)) = A ∪ (X – cl(A)). Since X – F is gsδI*-open, A ∪(X– cl(A)) is gsδI*-open.
Sufficiency-Let U = A ∪ (X – cl(A)). By hypothesis, U is gsδI*-open. Then X – U is gsδI*-closed and X – U = X – (A ∪ (X – cl(A))) = cl(A) ∩ (X – A) = cl(A) – A. Hence proved.

Theorem 3.10. Let (X, τ, I) be an ideal space. Then every subset of X is gsδI*-closed if and only if every semiopen subset of X is δI*-closed.
Proof. Necessity - Suppose every subset of X is gsδI*-closed. If U is a semiopen subset of X, then U is gsδI*-closed and so [U]* = U. Hence, U is δI*-closed.
Sufficiency - Suppose A⊆ U and U is semiopen. By hypothesis, U is δI*-closed. Therefore, [A]*⊆[U]* = U and hence A is gsδI*-closed.

Theorem 3.11. Let (X, τ, I) be an ideal space. If every subset of X is gsδI*-closed, then every open subset of X is δI*-closed.
Proof. Suppose every subset of X is gsδI*-closed. If U is an open subset of X, then U is gsδI*-closed and so [U]*⊆ U, since every open set is semiopen. Hence, U is δI*-closed.

Theorem 3.12. Intersection of a gsδI*-closed set and aδI*-closed set is always gsδI*-closed.
Proof. Let A be a gsδI*-closed set and G be any δI*-closed set of an ideal space (X, τ, I). Suppose A ∩ G ⊆ U and U is semiopen set in X. Then, A⊆ U ∪ (X – G). Now, X – G is δI*-open and hence open and so semiopen set. Therefore, U ∪ (X – G) is a semiopen set containing A. But A is gsδI*-closed and therefore, [A]*⊆ U ∪(X – G).

Therefore, $[A]^* \cap G \subseteq U$ which implies that, $[A \cap G]^* \subseteq U$. Hence, $A \cap G$ is gsδI*-closed.

Theorem3.13. In an ideal space (X, τ, I), for each $x \in X$, either $\{x\}$ is semiclosed or $\{x\}^c$ is gsδI*-closed.
Proof. Suppose that $\{x\}$ is not a semiclosed set, then $\{x\}^c$ is not a semiopen set and hence X is the only semiopen set containing $\{x\}^c$. Therefore, $[\{x\}^c]^* \subseteq X$ and hence $\{x\}^c$ is gsδI*-closed in (X, τ, I).

Theorem 3.14. Every gsδI*-closed, semiopen set is δI*-closed.
Proof. Let A be a gsδI*-closed, semiopen set in (X, τ, I). Since A is semiopen such that $A \subseteq A$, by hypothesis, $[A]^* \subseteq A$. Thus, A is δI*-closed.

Corollary 3.15. Every gsδI*-closed; open set is δI*-closed set.

Theorem 3.16. If A and B are gsδI*-closed sets in an ideal topological space (X, τ, I), then $A \cup B$ is a gsδI*-closed set in (X, τ, I).
Proof. Suppose that $A \cup B \subseteq U$, where U is semi open set in (X, τ, I). Then $A \subseteq U$ and $B \subseteq U$. Since A and B are gsδI*-closed sets in (X, τ, I), $[A]^* \subseteq U$ and $[B]^* \subseteq U$. Always, $[A \cup B]^* = [A]^* \cup [B]^*$. Therefore, $[A \cup B]^* \subseteq U$, whenever U is semi open. Hence, $A \cup B$ is gsδI*-closed set in (X, τ, I).

Theorem 3.17. Let (X, τ, I) be an ideal space. If A is a gsδI*-closed subset of X and $A \subseteq B \subseteq [A]^*$, then B is also gsδI*-closed.
Proof. The proof is clear.

Theorem 3.18. A subset A of an ideal space (X, τ, I) is gsδI*-closed if and only if $[A]^* \subseteq$ sker(A).
Proof. Necessity - Suppose A is gsδI*-closed and $x \in [A]^*$. If $x \notin$ sker(A), then there exist a semiopen set U such that $A \subseteq U$ but $x \notin U$. Since A is gsδI*-closed, $[A]^* \subseteq U$ and so $x \notin [A]^*$, a contradiction. Therefore, $[A]^* \subset$ sker(A).
Sufficiency - Suppose that $[A]^* \subseteq$ sker(A). If $A \subseteq U$ and U is semiopen then sker(A) $\subseteq$ U and so $[A]^* \subseteq U$. Therefore, A is gsδI*-closed.

Theorem3.19. Let A be a semi∧ - set of an ideal space (X, τ, I). Then A is gsδI*-closed if and only if A is δI*-closed.
Proof. Necessity - Suppose A is gsδI*-closed. Then by Theorem 3.18, $[A]^* \subseteq$ sker(A) = A, since A is semi∧ - set. Therefore, A is δI*-closed.
Sufficiency - The proof is follows from the Theorem 2.2.

Definition 3.20. A proper nonempty gsδI*-closed subset A of an ideal space (X, τ, I) is said to be maximal gsδI*-closed if any gsδI*-closed set containing A is either X or A.

Example 3.21. Let X = {a, b, c, d}, τ = {X, ϕ, {b}, {c, d}, {b, c, d}} and I = {ϕ, {d}}. Then {a, b, c} is a maximal gsδI*-closed set.

Theorem 3.22. In an ideal space (X, τ, I), the following are true.
(i) Let F be a maximal gsδI*-closed set and G be a gsδI*-closed set. Then $F \cup G = X$ or $G \subseteq F$.
(ii) Let F and G be maximal gsδI*-closed sets. Then $F \cup G = X$ or F = G.
Proof. (i)Let F be a maximal gsδI*-closed set and G be a gsδI*-closed set. If $F \cup G = X$, then there is nothing to prove. Assume that, $F \cup G \neq X$. Now, $F \subseteq F \cup G$. By Theorem 3.16, $F \cup G$
is a gsδI*-closed set. Since F is maximal gsδI*-closed, we have $F \cup G = X$ or $F \cup G = F$. Hence, $F \cup G = F$ and so $G \subseteq F$.
(ii) Let F and G be maximal gsδI*-closed sets. If $F \cup G = X$, then there is nothing to prove. Assume that, $F \cup G \neq X$. Then by (i), $F \subseteq G$ and $G \subseteq F$, which implies that, F = G.

Theorem 3.23. A subset A of an ideal space (X, τ, I) is gsδI*-open if and only if $F \subseteq [A]_{int}^*$ whenever F is semiclosed and $F \subseteq A$.
Proof. Necessity - Suppose A is gsδI*-open and F be a semiclosed set contained in A. Then $X - A \subseteq X - F$ and hence $[X - A]^* \subseteq X - F$. Thus, $F \subseteq X - [X - A]^* = [A]_{int}^*$.
Sufficiency - Suppose $X - A \subseteq U$, where U is semiopen. Then $X - U \subseteq A$ and X – U is semiclosed. Then $X - U \subseteq [A]_{int}^*$, which implies $[X - A]^* \subseteq U$. Therefore, X – A is gsδI*-closed and hence A is gsδI*-open.

Theorem 3.24. If A is a gsδI*-open subset of an ideal space (X, τ, I) and $[A]_{int}^* \subseteq B \subseteq A$. Then B is also a gsδI*-open subset of (X, τ, I).
Proof. Suppose $F \subseteq B$, where F is semiclosed set. Then, $F \subseteq A$. Since A is gsδI*-open, $F \subseteq [A]_{int}^*$. Since $[A]_{int}^* \subseteq [B]_{int}^*$, we have $F \subseteq [B]_{int}^*$. By the above Theorem 3.23, B is gsδI*-open.

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