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Relatively Prime Inverse Domination on Line Graph

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Abstract

Let G be non-trivial graph. A subset D of the vertex set $V(G)$ of a graph G is called a dominating set of G if every vertex in $V - D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set is called the domination number and is denoted by $\gamma(G)$. If $V - D$ contains a dominating set S of G, then S is called an inverse dominating set with respect to D. In an inverse dominating set S, every pair of vertices u and v in S such that $(deg(u), deg(v)) = 1$, then S is called relatively prime inverse dominating set. The minimum cardinality of a relatively prime inverse dominating set is called relatively prime inverse dominating number and is denoted by $\gamma_{rp}^{-1}(G)$. In this paper we find relatively prime inverse dominating number of some jump graphs.

Keywords: Domination number, Inverse domination number, Relatively prime domination number.

2020 AMS subject classifications: 05C69,05C76 [1]

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1 Introduction

By a graph, we mean a finite undirected graph with neither loops nor multiple edges. For graph theoretic terminology, we refer to the book by Chartrand and Lesniak [1]. All graphs in this paper are assumed to be non-trivial. In a graph $G = (V, E)$, the degree of a vertex v is defined to be the number of edges incident with v and is denoted by $deg(v)$. A set D of vertices of graph G is said to be a dominating set if every vertex in $V - D$ is adjacent to a vertex in D. A dominating set D is said to be a minimal dominating set if no proper subset of D is a dominating set. The minimum cardinality of a dominating set of a graph G is called the domination number of G and is denoted by $\gamma(G)$. Kulli V. R. et al. introduced the concept of inverse domination in graphs [8]. Let D be a minimum dominating set of G. If $V - D$ contains a dominating set S, then S is called a inverse domination set of G with respect to D. The inverse domination number $\gamma^{-1}(S)$ is the minimum cardinality taken over all the minimal inverse dominating set of G. The Jewel graph J_n is a graph with vertex set $V(J_n)$ = $\{u, x, v, y, v_i : 1 \leq i \leq n\}$ and edge set $E(J_n)$ = $\{ux, vx, uy, vy, xy, uv_i, vv_i : 1 \leq i \leq n\}$[7]. Bistar $B_{m,n}$ is the graph obtained by joining the center vertices of star graphs $K_{1,m}$ and $K_{1,n}$ by an edge. The vertex set of $B_{m,n}$ is $\{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ where u, v are apex vertices and u_i, v_i are pendent vertices. The edge set of $B_{m.n}$ is $\{uv, uu_i, vv_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $|V(B_{m,n})| = m+n+2, |E(B_{m,n})| = m + n + 1$[2]. A spider graph is a tree with at most one vertex of degree greater than 2[2]. Let P_n be a path graph with n vertices. The Comb graph is defined as $P_n \odot K_1$. It has $2n$ vertices and $2n - 1$ edges[3]. A wounded spider graph is a graph obtained by subdividing at most $n - 1$ edges of a star $K_{1,n}$. The wounded spider includes K_1, the star $K_{1,n-1}$[9]. A set $S \subseteq V$ is said to be relatively prime dominating set if it is a dominating set with at least two elements and for every pair of vertices u and v in S such that $(deg(u), deg(v)) = 1$. The minimum cardinality of a relatively prime dominating set of a graph G is called the relatively prime domination number of G and is denoted by $\gamma_{rpd}(G)$ [5]. The purpose of this paper is to study about the concept of relatively prime inverse domination on line graphs.

Definition 1.1. [6]*Let D be a minimum dominating set of a graph G. If $V - D$ contains a dominating set S of G, then S is called an inverse dominating set with respect to D. If every pair of vertices u and v in S such that $(deg(u), deg(v)) = 1$, then S is called relatively prime inverse dominating set. The minimum cardinality of a relatively prime inverse dominating set is called a relatively prime inverse domination number and is denoted by $\gamma_{rp}^{-1}(G)$. If the relatively prime inverse dominating set is absent, then $\gamma_{rp}^{-1}(G) = 0$.*

Definition 1.2. [4]*A line graph $L(G)$ of a simple graph G is obtained by associ-*

ating a vertex with each edge of the graph and connecting two vertices with an edge if only if the corresponding edges of G have a vertex in common.

Example 1.1. *Consider the graphs G and $L(G)$ which are given Figure 1. Clearly $\{e_1, e_4\}$ is a minimum dominating set of $L(G)$ and $\{e_2, e_5\}$ is a corresponding minimum inverse dominating set of $L(G)$ and $(deg(e_1), deg(e_4)) = (4, 3) = 1$ and so $\gamma_{rp}^{-1} L(G) = 2$.*

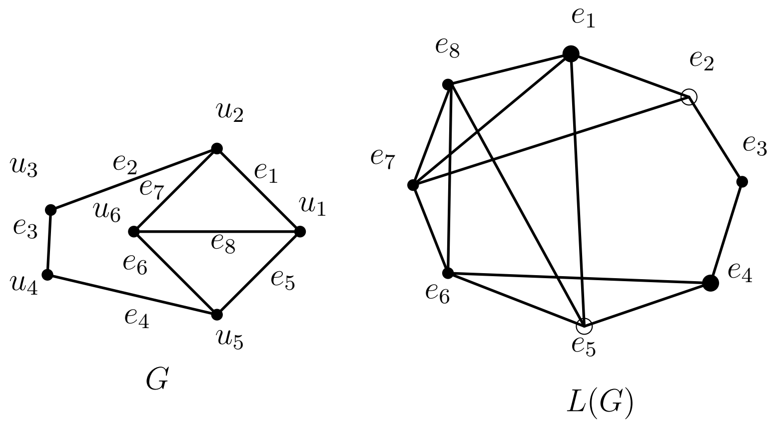

Figure 1: $G, L(G)$

We use the following theorem:

Theorem 1.1. [8] *For a path P_n, $\gamma_{rp}^{-1}(P_n)$ =* $\begin{cases} 2 & if \;\; 3 \le n \le 5 \\ 3 & if \;\; n = 6, 7 \\ 0 & otherwise \end{cases}$

2 Relatively prime inverse domination on line graph

Theorem 2.1. *For the spider graph $K_{1,n,n}$, $\gamma_{rp}^{-1}(L(K_{1,n,n}))$ = n.*

Proof. Let v be the centre vertex and the end vertices of $K_{1,n}$ be $v_1, v_2, ..., v_n$. Let $u_1, u_2, ..., u_n$ represent the vertices connected with $v_1, v_2, ..., v_n$, respectively. The resulting graph is the spider graph $K_{1,n,n}$ with vertex set $V(K_{1,n,n})$ = $\{v, v_i, v_i^{'} : 1 \le i \le n\}$ and $E(K_{1,n,n})$ = $\{vv_i, v_iv_i^{'} : 1 \le i \le n\}$. Clearly, $deg(v) = n$, $deg(v_i) = 2$, and $deg(v_i^{'}) = 1, 1 \le i \le n$. Let the line graph of the graph $K_{1,n,n}$ be $L(K_{1,n,n})$. Denote the edges vv_i by e_i and $v_iv_i^{'}$ by $e_i^{'}$. Clearly $V(L(K_{1,n,n})) = \{e_i, e_i^{'} : 1 \le i \le n\}$ and $E(L(K_{1,n,n})) = \{e_ie_j, e_ie_i^{'} : 1 \le i \ne j \le n\}$. Let D be a minimum dominating set of $L(K_{1,n,n})$ and S be a corresponding minimum inverse dominating set of $L(K_{1,n,n})$. Although $L(K_{1,n,n})$ contains n end vertices, any minimum dominating set of $L(K_{1,n,n})$ must include at least n vertices of $L(K_{1,n,n})$. Clearly, $D = \{e_i : 1 \leq i \leq n\}$ is a minimum dominating set and $S = \{e_i^{'} : 1 \leq i \leq n\}$ is the corresponding minimum inverse dominating set of $L(K_{1,n,n})$. Since $deg(e_i^{'}) = deg(e_j^{'}) = 1$ for $1 \leq i \neq j \leq n$, S is a minimum relatively prime inverse dominating set of $L((K_{1,n,n}))$. As a result, $\gamma_{rp}^{-1}(L(K_{1,n,n})) = n$. □

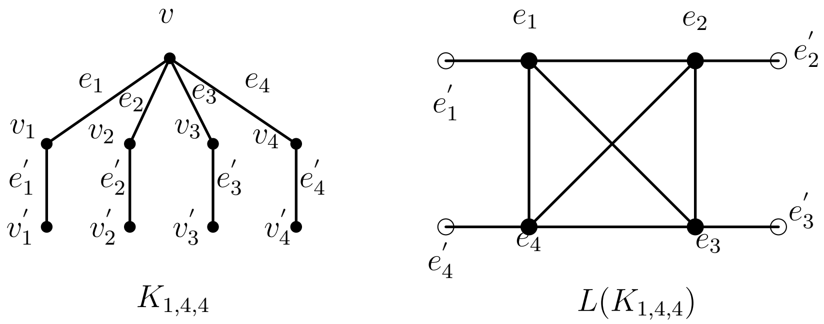

Figure 2: $K_{1,4,4}, L(K_{1,4,4})$

Theorem 2.2. *For the wounded spider graph $K_{1,n,s}$, $\gamma_{rp}^{-1}(L(G)) = s + 1$ where $s < n$.*

Proof. Let v be the centre vertex and $v_1, v_2, ..., v_n$ be the end vertices of $K_{1,n}$. Attach $u_1, u_2, ..., u_s$ to $v_1, v_2, ..., v_n$ as appropriate where $s < n$. The resulting graph is the wounded spider graph $K_{1,n,s}$ with vertex set $V(K_{1,n,s}) = \{v, v_i, u_j : 1 \leq i \leq n, 1 \leq j \leq s\}$ and $E(K_{1,n,s}) = \{vv_i, v_ju_j : 1 \leq i \leq n, 1 \leq j \leq s\}$. Clearly in $K_{1,n,s}$, $deg(v) = n, deg(v_i) = 2, 1 \leq i \leq s, deg(v_k) = 1, s+1 \leq i \leq n$ and $deg(u_i) = 1$, $1 \leq i \leq s$. Let the line graph of the graph $K_{1,n,s}$ be $L(K_{1,n,s})$ where we denote the edge vv_i by e_i and v_ju_j by $e_j^{'}, 1 \leq i \leq n, 1 \leq j \leq s$. Clearly, $V(L(K_{1,n,s})) = \{e_i, e_j^{'} : 1 \leq i \leq n, 1 \leq j \leq s\}$ and $E(L(K_{1,n,s})) = \{e_ie_k, e_je_j^{'} : 1 \leq i \neq k \leq n, 1 \leq j \leq s\}$. Also in $L(K_{1,n,s})$, $deg(e_j) = n, deg(e_j^{'}) = 1$ and $deg(e_i) = n - 1$, $1 \leq j \leq s$ and $s + 1 \leq i \leq n$. Let D be a minimum dominating set of $L(K_{1,n,s})$ and S be a minimum inverse dominating set of $L(K_{1,n,s})$ with respect to D. Since $L(K_{1,n,s})$ contains s end vertices, any minimum dominating set of $L(K_{1,n,s})$ must include at least s vertices of $L(K_{1,n,s})$. Clearly, $D = \{e_j : 1 \leq j \leq s\}$ and $S = \{e_n, e_j^{'} : 1 \leq j \leq s\}$ is a corresponding minimum inverse dominating set of $L(K_{1,n,s})$. Since the degree sequence of vertices in S is $(n, 1, 1, ..., 1)$, S is a minimum relatively prime inverse dominating set of $L((K_{1,n,s}))$ and hence $\gamma_{rp}^{-1}(L(K_{1,n,s})) = s + 1$. □

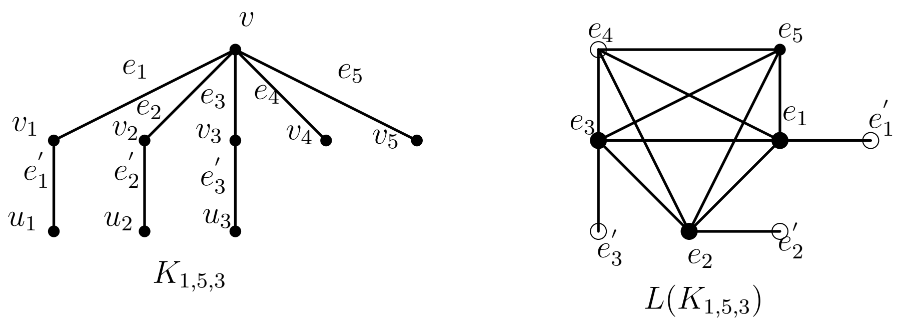

Figure 3: $K_{1,5,3}, L(K_{1,5,3})$

Theorem 2.3. *For the jewel graph J_n, $\gamma_{rp}^{-1}(L(J_n))$ = 2 if $n \geq 1$.*

Proof. Consider a 4-cycle $xwyux$. Join x and y. Now adding n new vertices $v_i, 1 \leq i \leq n$. Join v_i with u and $w, 1 \leq i \leq n$. The resulting grph is the jewel graph J_n with vertex set $V(J_n)$ = $\{x, y, u, w, v_i : 1 \leq i \leq n\}$ and edge set $E(J_n) = \{e_i, e_j^{'}, e_j^{''} : 1 \leq i \leq 5, 1 \leq j \leq n\}$, where $e_1 = xw, e_2 = wy, e_3 = yu, e_4 = ux, e_5 = xy, e_j^{'} = uv_j, e_j^{''} = wv_j$. Let the line graph of J_n be $L(J_n)$ where $V(L(J_n)) = E(J_n) = \{e_i, e_j^{'}, e_j^{''} : 1 \leq i \leq 5, 1 \leq j \leq n\}$ and $E(L(J_n)) = \{e_ie_{i+1}, e_1e_4, e_5e_j, e_j^{'}e_i, e_j^{'}e_k^{'}, e_j^{''}e_m, e_j^{''}e_p^{''} : 1 \leq i \leq 3, 1 \leq j \leq k, p \leq n, 3 \leq l \leq 4, 1 \leq m \leq 2\}$, $i \neq k$. Let D be a minimum dominating set of $L(J_n)$ and S be a corresponding minimum inverse dominating set. In $L(J_n)$, the number of vertices is $2n + 5$ and the maximum degree is $2n - 1$ and so any minimum dominating set contains at least two vertices. Now e_1 is adjacent to all vertices except e_3 and $e_i^{'}$, $1 \leq i \leq n$; $e_1^{'}$ is adjacent to $e_3, e_4, e_1^{''}$ and $e_i^{'}, 2 \leq i \leq n$. Hence $D = \{e_1, e_1^{'}\}$ is a minimum dominating set of $L(J_n)$. Clearly, $S = \{e_3, e_1^{''}\} \subseteq V - D$ is also a minimum dominating set of $L(J_n)$. Hence, S is a minimum inverse dominating set of $L(J_n)$. In $L(J_n)$, $deg(e_3) = n + 3$, $deg(e_1^{''}) = n + 2$ and therefore $(deg(e_3), deg(e_1^{''}))$ = $(n + 3, n + 2) = 1$. This implies that S is a minimum relatively prime inverse dominating set of $L(J_n)$ and so $\gamma_{rp}^{-1}(L(J_n)) = 2$. □

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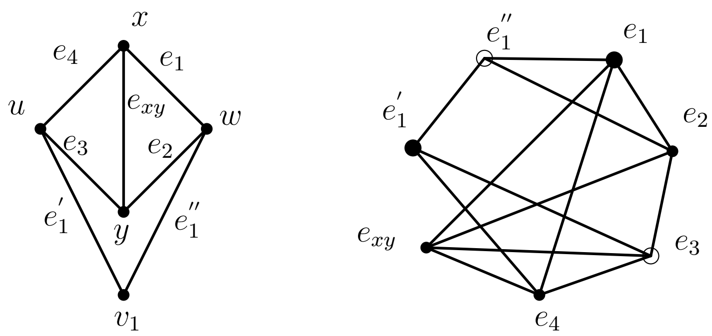

J_1 $L(J_1)$

Figure 4: $J_1, L(J_1)$

Theorem 2.4. *For the bistar graph $B_{m,n}$, $\gamma_{rp}^{-1}(L(B_{m,n}))$ =* $\begin{cases} 2\ if\ (m,n) = 1 \\ 0\ otherwise \end{cases}$.

Proof. A bistar graph $B_{m,n}$ consists of two star graphs $K_{1,m}$ and $K_{1,n}$ having center vertices u_0 and v_0 respectively. Join u_0 and v_0 with an edge. The resulting graph is a bistar graph $B_{m,n}$ with the vertex set $V(B_{m,n})$ = $\{u_i, v_j : 0 \leq i \leq m, 0 \leq j \leq n\}$ and edge set $E(B_{m,n})$ = $\{u_0u_i, v_0v_j, u_0v_0 : 1 \leq i \leq m, 1 \leq j \leq n\}$. Let the line graph of $B_{m,n}$ be $L(B_{m,n})$ with the vertex set $V(L(B_{m,n}))$ = $\{e_0, e_i, e_j^{'} : 1 \leq i \leq m, 1 \leq j \leq n\}$ where $e_0 = u_0v_0, e_i = u_0u_i, e_j^{'} = v_0v_j$ and edge set $E(L(B_{m,n}))$ = $\{e_0e_i, e_0e_j^{'}, e_ie_k, e_j^{'}e_l^{'} : 1 \leq i \neq k \leq m, 1 \leq j \neq l \leq n\}$. Clearly in $L(B_{m,n})$, $deg(e_0) = m + n, deg(e_i^{'}) = m$ and $deg(e_j^{'}) = n, 1 \leq i \leq m, 1 \leq j \leq n$. Let D be a minimum dominating set of $L(B_{m,n})$ and S be a corresponding minimum inverse dominating set of $L(B_{m,n})$. In $L(B_{m,n})$, the vertex e_0 dominates all other vertices and so the unique minimum dominating set of is $D = \{e_0\}$. In $V - D$, each e_x dominates e_0 and all other $e_i, 1 \leq i \leq m$ and $i \neq x$ and also each $e_y^{'}$ dominates all other $e_j^{'}, j \neq y$ and $1 \leq j \leq n$. Hence a minimum inverse dominating set $S = \{e_x, e_y^{'}\}$ for some x, y where $1 \leq x \leq m, 1 \leq y \leq n$. Now in $L(B_{m,n})$, $(deg(e_x), deg(e_y^{'})) = (m, n)$. This implies that S is a minimum relatively prime inverse dominating set if and only if $(m, n) = 1$. Hence the proof.

□

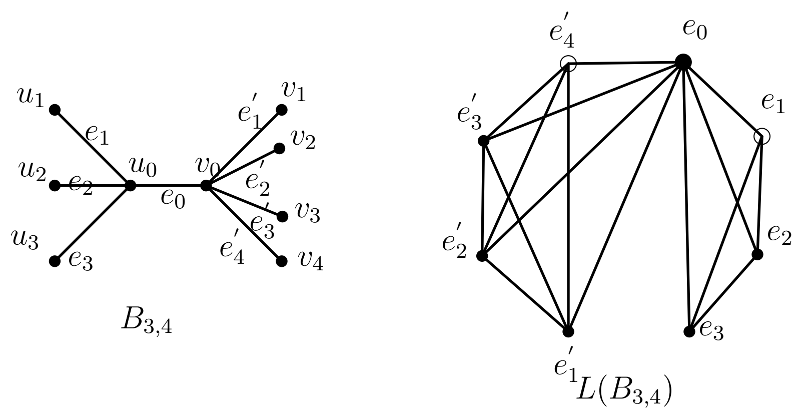

Figure 5: $B_{3,4}, L(B_{3,4})$

Theorem 2.5. *For the comb graph $P_n \odot K_1$,*

$$\gamma_{rp}^{-1}(L(P_n \odot K_1)) = \begin{cases} 2 \; if \; n = 2,3 \\ 3 \; if \; n = 4,5 \\ 0 \; otherwise \end{cases}$$

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Proof. Consider the path $P_n = v_1 v_2 .... v_n$. For $1 \leq i \leq n$, add vertex u_i which is adjacent to v_i. The resulting graph $G = P_n \odot K_1$ is a comb graph with vertex set $V(G) = \{v_i, u_i : 1 \leq i \leq n\}$ and edge set $E(G) = \{e_i, e_j' : 1 \leq i \leq n-1, 1 \leq j \leq n\}$ where $e_i = v_i v_{i+1}, e_j' = v_j u_j, 1 \leq i \leq n-1, 1 \leq j \leq n$. Let the line graph of comb graph G be $L(G)$ where the vertex set $V(L(G)) = E(G) = \{e_i, e_j' : 1 \leq i \leq n-1, 1 \leq j \leq n\}$ and edge set $E(L(G)) = \{e_i e_{i+1}, e_j e_j', e_j e_{j+1}' : 1 \leq i \leq n-2, 1 \leq j \leq n-1\}$. Clearly in $L(G)$, $deg(e_i) = 4, 2 \leq i \leq n-2, deg(e_1) = deg(e_{n-1}) = 3, deg(e_j') = 2, 2 \leq j \leq n-1$ and $deg(e_i) = deg(e_n') = 1$. Let D be a minimum dominating set of $L(G)$ and S be a corresponding minimum inverse dominating set of $L(G)$. Now we cosider the following five cases.

Case 1. $n = 2$

Then $L(G)$ is P_3. By Theorem 1.1, $\gamma_{rp}^{-1}(L(G)) = 2$.

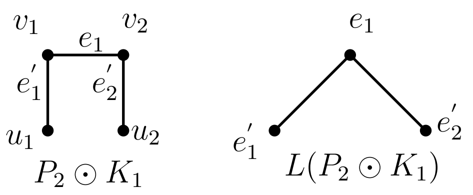

Figure 6: $P_2 \odot K_1$, $L(P_2 \odot K_1)$

Case 2. $n = 3$

In $L(G), e_1$ is adjacent to all vertices except $e_3^{'}$, $e_3^{'}$ is adjacent to e_2 only. Hence, $D = \{e_1, e_3^{'}\}$ is a minimum dominating set of $L(G)$ and a corresponding minimum inverse dominating set $S = \{e_2, e_1^{'}\}$. In $L(G)$, $(deg(e_2), deg(e_1^{'}))$ = $(3, 1) = 1$. This implies that S is a minimum relatively prime inverse dominating set of $L(G)$ and so $\gamma_{rp}^{-1}(L(G)) = 2$.

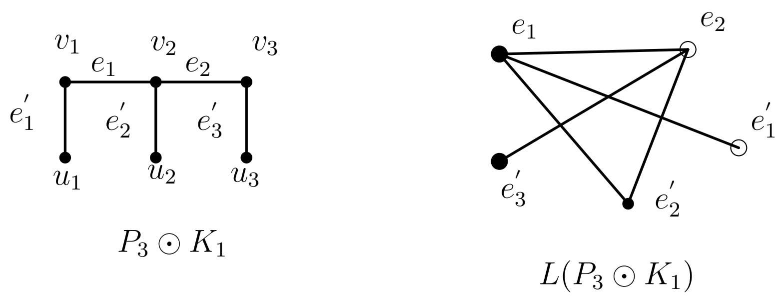

Figure 7: $P_3 \odot K_1$, $L(P_3 \odot K_1)$

Case 3. $n = 4$

In $L(G), e_1$ is adjacent to all vertices except e_3 and $e_i^{'}, i = 3, 4$; $e_3^{'}$ is adjacent to all vertices except $e_1, e_i^{'}, i = 1, 2$. Hence, $D = \{e_1, e_3\}$ is a minimum dominating set of $L(G)$ and a corresponding minimum inverse dominating set $S = \{e_2, e_1^{'}, e_4^{'}\}$. The degree sequence vertices in S is (4,1,1). This implies that S is a minimum relatively prime inverse dominating set of $L(G)$ and so $\gamma_{rp}^{-1}(L(G)) = 3$.

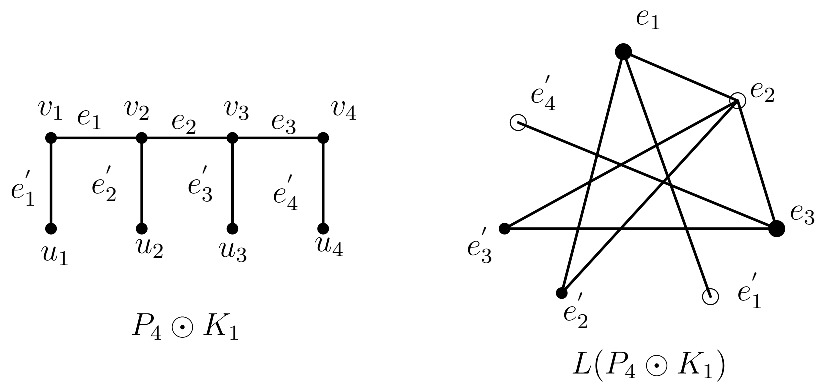

Figure 8: $P_4 \odot K_1$, $L(P_4 \odot K_1)$

Case 4. $n = 5$

In $L(G), e_1$ is adjacent to all vertices except $e_i, i = 3, 4, e_1', e_4', e_5'$; e_3 is adjacent to all vertices except e_1, e_1', e_2', e_5'; e_5' is adjacent to all vertices except e_4. Hence, $D = \{e_1, e_3, e_5'\}$ is a minimum dominating set and a corresponding minimum inverse dominating S = $\{e_2, e_4, e_1'\}$. In $L(G)$, the degree sequence of vertices in S is $(4, 3, 1)$. This implies that S is a minimum relatively prime inverse dominating set of $L(G)$ and so $\gamma_{rp}^{-1}(L(G)) = 3$.

Case 5. $n \geq 6$

The degree sequence of $L(G)$ is $\{4, 4, \cdots, 4(n-3)$times$, 3, 3, 2, 2, \cdots, 2(n-2)$times$, 1, 1\}$. Any minimum dominating set must contain at least four vertices and so any minimum inverse dominating set S as at least four vertices of different degrees and thereby, there exists a pair of vertices (x, y) in S such that $(deg(x), deg(y)) = 2$ or 4. Hence, $\gamma_{rp}^{-1}(L(G)) = 0$.

Thus the theorem following five cases. □

3 Conclusion

Inspired by inverse dominating set and relatively prime dominating set, we introduce the relatively prime inverse domination number on line graph. We have determined the relatively prime inverse domination on line graph of some standard graphs like spider graph, wounded spider graph, jewel graph, bistar graph, and comb graph. Furthermore our results are also justified with suitable examples.

The relatively prime inverse domination number can be obtained for many more graphs.

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