Ratio Mathematica Volume 47, 2023

Door spaces on N- Topology

Loyala Foresith Spencer J *
Davamani Christober M†

Abstract

In this article, we explore the idea of door space on N-topological space. Here, we discuss which door spaces in this space are related with $N\tau$ submaximal. The equivalent conditions shows how it connects a N-topological property. Also, we derive various door spaces using separation axioms and discuss the characteristics of such door spaces. We take a strong form of open set in N-topological space and introduce a new door space called $N\tau\beta$ - door space. In addition, we analyze $N\tau\beta$-door space and discuss the relationship between a $N\tau\beta$-locally closed set and $N\tau$-closed set.

Keywords: N-topology; door space; sub-maximal; $N\tau_d\beta$-open sets.
2020 AMS subject classifications: 54A05, 54A10, 54C05 [1]

*The American College, Madurai, India; e-mail: spencerjraja@gmail.com.
†The American College, Madurai, India; e-mail:christober.md@gmail.com.

[1]Received on September 15, 2022. Accepted on March 10, 2023, Published on April 4, 2023. DOI: 10.23755/rm.v39i0.957. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

In 1955, J.L. Kelly [2], introduced the term 'door' in classical topology. He investigated the relationship between the different topological spaces and door spaces. McCartan [5] found three types of door spaces and established the concepts of connected door spaces and maximally connected. Muckenhoupt and Williams [6] concluded that there exists a non-zero Borel measure in every connected door space. Mathew [4]enquired about hyper-connected door space and proposed its related concepts. In 1983, Monsef et al.[1] initiated the study of β-open sets and β-continuity in a topological space.

This article explores the idea of door space in N - topological space. Here, we introduce $N\tau$-door space and discuss which door spaces are related with $N\tau$ submaximal. The equivalent conditions shows how it connects with $N\tau$ frontier, a N-topological property. Also, we derive various door spaces using separation axioms and discuss the characteristics of such door spaces. We take a strong form of open set in N-topological space and introduce a new door space called $N\tau\beta$ - door space . Finally, we introduce $N\tau\beta$ - locally closed sets and its properties which are all the essential tool for the future development of this concept.

2 $N\tau$ - door spaces

In this section, we establish various door spaces in N-topological space and explicit its properties. Moreover, discussed about the sub maximal concept and analyze its characterization in this space.

Definition 2.1. *A $(P, N\tau)$ - space is called Hausdorff space of N topological space if for every given pair of different points $z, q \in P$,there exists $R, S \in N\tau$ $O(P)$ such that $z \in R$, $q \in S$, $R \cap S = \phi$ and is denoted by $N\tau$ - Hausdorff space.*

Definition 2.2. *A $(P, N\tau)$ - space is called semi - hausdorff space of N topological space if for every given pair of different points $z, q \in P$,there exists $R, S \in N\tau$ $SO(P)$ such that $z \in R$, $q \in S$, $R \cap S = \phi$ and is denoted by $N\tau$ - semi hausdorff space.*

Theorem 2.1. *Each $N\tau$ hausdorff space of $(P, N\tau)$ is $N\tau$ - semi hausdorff space.*

Proof. Suppose $(P, N\tau)$ is a $N\tau$ hausdorff space then there exisits two disjoint points which can be isolated by disjoint $N\tau$ open set. Since each $N\tau$ open set is a $N\tau$ - semi open then we may conclude that each $N\tau$ hausdorff space of $(P, N\tau)$ is $N\tau$ - semi hausdorff space. □

Remark 2.1. *The inverse of the theorem 5.2.3 need not be true which is shown below.*
Let P = {l, m, n, o, p}. For $N = 3$, consider $\tau_1 = \{\phi, \{l\}, \{l, p\}, \{l, m, o, p\}, P\}$, τ_2 = $\{\phi, \{l, m, o\}, \{m, o\}, P\}$ and τ_3 = $\{\phi, \{m, o, p\}, \{l, m, o, p\}, P\}$.
Then 3τ = $\{\phi, \{l\}, \{p\}, \{l, p\}, \{m, o\}, \{l, m, o\}, \{m, o, p\}, \{l, m, o, p\}, P\}$. Here $\{l, n\}$ is 3τ - semi open set but not 3τ - open.

Definition 2.3. *If every subset of $(P, N\tau)$ is either $N\tau$ - open or $N\tau$-closed then $(P, N\tau)$ is called as $N\tau$ - door space.*

Example 2.1. *Let $S = \{x, y, z\}$. For $N = 2$, $\tau_1 = \{\phi, \{z\}, \{x, z\}, S\}$. Then 2τ $O(S) = \{\phi, \{x\}, \{z\}, \{x, z\}, S\}$, 2τ $C(S)$ = $\{\phi, \{y, z\}, \{x, y\}, \{y\}, S\}$. Therefore S is a 2τ - door space.*

Theorem 2.2. *Every subspace G of a $N\tau$ - door space is $N\tau$ - door space.*

Proof. Let $(P, N\tau)$ be a door space. Let $G \subseteq P$ and $U \in G$. Since P is a $N\tau$ door space then U is either $N\tau$ open or $N\tau$ closed in P and hence in G. Therefore G is also a $N\tau$ door space. □

Definition 2.4. *A subset S of $(P, N\tau)$ is $N\tau$ - dense if $N\tau$ - cl$(S) = P$.*

Definition 2.5. *A N-topological space $(P, N\tau)$ is sub maximal if every $N\tau$ - dense subset of P is $N\tau O(P)$ and is denoted as $N\tau$- sub maximal.*

Theorem 2.3. *Every door space $(P, N\tau)$ is $N\tau$ - sub maximal.*

Proof. Let $(P, N\tau)$ be a door space and $V \subset P$ be a $N\tau$ dense. If V is not a $N\tau$ open then it is $N\tau$ closed since P is a $N\tau$ door space. Now $V = N\tau cl(V) = P$ and V is $N\tau$ open. Hence P is $N\tau$ - sub maximal. □

Theorem 2.4. *Every subspace L of a sub maximal space $(P, N\tau)$ is again a $N\tau$ - sub maximal.*

Proof. Let A be a $N\tau$ dense subset of L. Then $N\tau cl(A) \cap L = L$ and so $L \subset N\tau cl(A)$. Since $A \cup (P - N\tau cl(A))$ is $N\tau$ dense in P then it is an $N\tau$ open subset of P. Hence $S \cap (A \cup (P - N\tau cl(A))) = A$ is $N\tau$ - open in L or equivalently L is $N\tau$ - submaximal. □

Theorem 2.5. *In $(P, N\tau)$, the following conditions are equivalent*

1. *P is $N\tau$ - submaximal.*

2. *For any $A \subset P$, the subspace $N\tau$ - $Fr(A)$ = $N\tau$-cl$(A)-$ $N\tau$ - int(A) = $N\tau$-cl$(A) \cap N\tau$-cl$(P - A)$ is discrete.*

Proof. (1) $\implies$ (2) Let $y \in N\tau Fr(A)$. Since A is $N\tau$ dense in $N\tau cl(A)$ then so is $A \cup \{y\}$. Since $N\tau cl(A)$ is sub maximal according to theorem 5.2.11 then $A \cup \{y\} = N\tau cl(A) \cap U$ where U is $N\tau$ open in P. In the same way it can seen that $A^c \cup \{y\} = N\tau cl(A^c) \cap V$ where V is $N\tau$ open in P. Thus $\{y\} = (A \cup \{y\}) \cap (A^c \cup \{y\}) = N\tau cl(A) \cap N\tau cl(P - A) \cap U \cap V$. Hence $\{y\}$ is $N\tau$ open in $N\tau Fr(A)$ and so $N\tau Fr(A)$ is discrete.
(2) $\implies$ (1) Let A be a $N\tau$ dense in P. By assumption, $N\tau cl(A) - N\tau int(A) = P - N\tau int(A)$ is discrete and thus $A - N\tau int(A)$ is its $N\tau$ - open subset. Hence $A - N\tau int(A) = (P - N\tau int(A)) \cup U$ where U is $N\tau$ open in P. Thus $A - N\tau int(A) \subset U$ and so $A - N\tau int(A) \subset U - N\tau int(A)$. For the reverse inclusion if $y \in U - N\tau int(A)$ then $y \in (P - N\tau int(A)) \cap U = A - N\tau int(A)$. This shows that $A - N\tau int(A) = U - N\tau int(A)$ and hence $A = U \cup N\tau int(A)$. Thus A is $N\tau$ open in P. □

Theorem 2.6. *In $(P, N\tau)$, the following conditions are equivalent*

1. *P is $N\tau$ - submaximal.*
2. *Every $N\tau$ - pre open subset of P is $N\tau$ - open.*

Proof. (1) $\implies$ (2) Let A be a $N\tau$ pre open in P. Then $A \subset N\tau int(N\tau cl(A))$. Since A is $N\tau$ dense in $N\tau cl(A)$ and $N\tau cl(A)$ is $N\tau$ sub maximal according to theorem 5.2.11 then A is $N\tau$ open in $N\tau cl(A)$. Thus A is $N\tau$ open in $N\tau int(N\tau cl(A))$. Since $N\tau int(N\tau cl(A))$ is $N\tau$ open in P then we may conclude that A is $N\tau$ open in P.
(2) $\implies$ (1) Let A be a $N\tau$ dense in P. Since $A \subset P = N\tau int(P) = N\tau int(N\tau cl(P))$ then A is $N\tau$ - pre open and by assumption, $N\tau$ open. This shows that P is $N\tau$ - sub maximal. □

Theorem 2.7. *If $(P, N\tau)$ is submaximal and $U \subset P$ then U is $N\tau$ - open iff it is the intersection of a $N\tau$ - dense and $N\tau$ - regular open [3].*

Proof. It is enough to prove that for every $N\tau$ open set U, we have $U = D \cap V$ where D is $N\tau$ dense and V is $N\tau$ regular open since the reverse inclusion is trivial. Clearly $U \subset N\tau int(N\tau cl(U))$. Thus $U = N\tau cl(U) - (N\tau cl(U) - U)$ $= N\tau cl(U) \cap (P - (N\tau cl(U) - U)) = N\tau int(N\tau cl(U)) \cap (U \cup P - N\tau cl(U))$ where $N\tau int(N\tau cl(U)) = V$ is $N\tau$ - regular open and $U \cup (P - N\tau cl(U)) = D$ is $N\tau$ - dense. □

Theorem 2.8. *The homeomorphic image of $N\tau$ - door space is a $N\sigma$ - door space.*

Proof. Consider $(Z, N\tau)$ and $(Q, N\sigma)$ are door spaces and $\eta\colon Z \to Q$ be a homeomorphism. Let $U \subseteq Q$. Consider $\eta^{-1}(U) \subseteq Z$. Since Z is $N\tau$ - door space then η^{-1} is either $N\tau$ - open or $N\tau$ - closed in Z. Now $\eta(\eta^{-1}(U)) = U$ and U is either $N\sigma$- open or $N\sigma$ - closed in Q. □

Definition 2.6. *In $(P, N\tau)$, if every subset of P is either $N\tau$ semi-open or $N\tau$ semi-closed then $(P, N\tau)$ is semi-door space and is denoted by $N\tau$ semi-door space*

Theorem 2.9. *The homeomorphic image of $N\tau$ semi-door space is a $N\sigma$ semi-door space.*

Proof. Let $(Z, N\tau)$ and $(Q, N\tau)$ are N - topological spaces and $(Z, N\tau)$ be $N\tau$ semi-door space. A mapping $\eta\colon Z \to Q$ be a homeomorphism. Let $P \subset Q$. Consider $\eta^{-1}(P) \subset Z$, since Z is $N\tau$ semi - door space then $\eta^{-1}(P)$ is either $N\tau$ semi -open or $N\tau$ semi - closed. So $\eta(\eta^{-1}(P)) = P$ is either $N\tau$ - semi open or $N\tau$ - semi closed. Hence Q is a $N\tau$ - semi door space. □

Theorem 2.10. *A $N\tau$ - clopen subspace of a $N\tau$ semi-door space is $N\tau$ semi-door space.*

Proof. Let $(Z, N\tau)$ be semi-door space. Let Q be a $N\tau$-clopen subset of Z. Let $A \subseteq Q$ and $A \subseteq Z$. Since Z is a $N\tau$ semi-door space, then A is either $N\tau$ semi open or $N\tau$ semi closed in Z. Since Q is $N\tau$ open and $N\tau$ closed, then A is either $N\tau$ semi open or $N\tau$ semi closed in Q. Hence Q is $N\tau$ semi-door space. □

Theorem 2.11. *If $(P, N\tau)$ be a door space and if $z \in P$, S is a $N\tau$ - neighbourhood of z, then $S - z \in N\tau$ and $S \in N\tau$.*

Proof. Let S be an neighbourhood of a point v and if v is $N\tau int(S)$ then it is enough to prove that $S - v$ is $N\tau$ open. If we assume $S - v$ is not $N\tau$ - open then $P - (S - v) = (P - S) \cup v$ should be $N\tau$ open. This contradicts that $v = S \cap ((P - S) \cup v)$ should be $N\tau$ - open. □

Definition 2.7. *In $(P, N\tau)$, if every two disjoint points in P can be isolated by disjoint $N\tau$ open sets then $(P, N\tau)$ is a hausdorff door space and is denoted by $N\tau$-hausdorff door space.*

Definition 2.8. *In $(P, N\tau)$, if every two disjoint points in P can be isolated by disjoint $N\tau$ semi open sets then $(P, N\tau)$ is a semi - hausdorff door space and is denoted by $N\tau$ semi-hausdorff door space.*

Definition 2.9. *A $N\tau$ - semi door space is said to be hausdorff semi-door space of $(P, N\tau)$ if a given pair of different points $r, s \in P$, there exist $M, N \in N\tau O(P)$ such that $r \in M$, $s \in N$, $M \cap N = \phi$ and is denoted by $N\tau$-hausdorff semi door space.*

Proposition 2.1. *Every $N\tau$ - hausdorff door space is $N\tau$ semi-hausdorff door space.*

Proposition 2.2. *If $(P, N\tau)$ be hausdorff door space and $z \in P$ then $r \in N\tau$ $SO(P) \Longleftrightarrow r \in N\tau$.*

Proposition 2.3. *If $(P, N\tau)$ be hausdorff door space and $Z, Q \in (P, N\tau)$. If $z \in N\tau SO(P)$ and $Q \in N\tau$ then $Z \cap Q \in N\tau$ $SO(P)$.*

Theorem 2.12. *$N\tau$ - semi-hausdorff door space has atmost one limit point.*

Proof. Consider $(P, N\tau)$ be a hausdorff space. Let a, b are distinct limit points in P. Since P is $N\tau$ - semi-hausdorff, $\exists\ G, H \in SO(P)$: $a \in G$, $b \in H$ and $G \cap H = \phi$. Since P is $N\tau$ - door space then $U = \{G - \{a\}\} \cup \{b\}$ is either $N\tau$ - open or $N\tau$ - closed. Suppose if it is $N\tau$ - open then by theorem 5.2.19 $U \cap H = \{b\}$ is $N\tau$ - semi open and hence by theorem 5.2.20 $U \cap H$ is $N\tau$ - open. Otherwise U^c is $N\tau$ - open and $U^c \cap G = \{a\}$ is $N\tau$ - semi open and $N\tau$ - open by propositions 5.2.24 and 5.2.25. Hence at least one of the two point will be isolated in P and by contradiction the result is proved. □

Proposition 2.4. *A $N\tau$ - hausdorff semi-door space has atmost one limit point.*

Proof. Proof is similar as discussed in the previous result. □

3 $N\tau\beta$-door space and locally $N\tau\beta$-closed set

In this section, we introduce and analyze $N\tau\beta$ - door space and $N\tau\beta$ - locally closed sets.

Definition 3.1. *A subset U of P is said to be locally $N\tau$-closed set if $U = R \cap S$ where R is a $N\tau$ - closed in P. The set of all locally $N\tau$ - closed sets are denoted by $LN\tau$-$Cl(P)$*

Definition 3.2. *A subset U of P is said to be locally $N\tau\beta$-closed set if $U = R \cap S$ where R is a $N\tau\beta$ - open and S is $N\tau\beta$-closed in P. The set of all locally $N\tau\beta$ - closed sets are denoted by $LN\tau\beta$-$Cl(P)$*

Definition 3.3. *If every subset of $(P, N\tau)$ is either $N\tau\beta$-open or $N\tau\beta$ - closed then $(P, N\tau)$ is called $N\tau\beta$ - door space and signified by $N\tau\beta_d$.*

Example 3.1. *Let $S = \{x, y, z\}$. For $N = 2$, $\tau_1 = \{\phi, S\}$, $\tau_2 = \{\phi, \{x, y\}, S\}$. Then 2τ $O(S) = \{\phi, \{x, y\}, S\}$, $2\tau Cl(S) = \{\phi, \{z\}, S\}$. Here S is a $2\tau\beta$ - door space. $LN\tau$-$Cl(S) = \{\phi, \{x, y\}, \{z\}, S\}$ and $LN\tau\beta$-$Cl(P) = \{\phi, \{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{z, x\}, S\}$.*

Remark 3.1. *From the example 5.3.4, we get*

1. *Every locally $N\tau$-closed set is locally $N\tau\beta$- closed set but its converse need not be true.*
2. *Every $N\tau$-door space is a $N\tau\beta_d$ but its converse need not be true.*

Remark 3.2. *If $(P, N\tau)$ is $N\tau\beta$ - door space then $LN\tau\beta$-$\mathbb{C}l(P) = \wp(P)$.*

Theorem 3.1. *Let $G \subseteq P$. Then the following are equivalent:*

1. *G is locally $N\tau\beta$ - closed sets*
2. *$G = F \cap N\tau$-$cl(G)$ for some $N\tau\beta$-open set F.*

Proof. (1) $\implies$ (2) Let $G \in LN\tau\beta cl(P)$. Now there will be a $N\tau\beta$ - open set F and a $N\tau\beta$ - closed subset E such that $G = F \cap E$ since $G \subseteq F$ and $G \subseteq N\tau\beta cl(P)$ then $N\tau\beta cl(P) \subseteq E$. Hence $F \cap N\tau\beta cl(P) \subseteq F \cap E = G$. Therefore $G = F\cap N\tau\beta cl(G)$. This proves (1) $\implies$ (2) .
(2) $\implies$ (1) By definition, $N\tau\beta cl(G)$ is $N\tau\beta$ - closed. $\therefore G = F\cap LN\tau\beta cl(G) \in N\tau\beta LC(P)$. □

4 Conclusion

In this paper, we introduced the idea of door spaces in N-topological space. Their structural properties have been discussed and emphasized. Some of the important results arrived through illustrated examples. The importance is to analyse the relationship with other N- topological properties. So, We investigated submaximal concepts and locally closed sets through N-topology. In addition, we introduce $N\tau\beta$-door space and discuss the relationship between a $N\tau\beta$-locally closed set and $N\tau$-closed set. With the help of these locally closed sets it can be extend to introduce locally continuous maps in this topological space. These prime ideas can open the future scope of this concept and extended to other research areas of topology such as Fuzzy topology, Digital topology, and so on.

References

[1] Abd el-monsef, M.E., El-deeb S.N. and Mahmoud R.A. β-open sets and β-continuous mappings, *Bull. Fac. Sci.,* 12:77 – 90,1983.

[2] Kelly J.L. General topology, *Princeton, NJ. D. Van Nastrand*, 1955.

[3] Loyala F. Spencer J. and Davamani Christober M. Theta open sets in N-topology, *Ratio Mathematica*, 43:163 – 175, 2022.

[4] Mathew P.M. On hyper connected spaces, *Indian j. pure appl. math.*.1988.

[5] McCartan S.D. Door Spaces are identifiable, *Proceedingds of Roy irish acad. sect. A*, 87 (1): 13 – 16, 1987.

[6] Muckenhoupt B. and Williams V. Borel measures on connected Door Spaces, *Riv. Mat. Univ. Parma.,* 3(2): 103 – 108, 1973.