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The Connected Vertex Strong Geodetic Number of a Graph

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Abstract

In this paper we introduce the concept of connected vertex strong geodetic number $cg_{sx}(G)$ of a graph G at a vertex x and investigate its properties. We determinebounds for it and find the same for some special classes of graphs. We prove that$sg_x(G) \leq csg_x(G)$ for any vertex x in $G$is connected graphs of order $n \geq 2$with one are characterized for some vertex x in G.Necessary conditions for$sg_x(G)$ to be n or $n-1$ are given for some vertex x in G. It is shown for every pair of integers$a$ and b with $2 \leq a \leq b$, there exists a connected graph G such that $sg_x(G) = a$ and $csg_x(G) = b$ for some vertex x in G.

Keywords: strong geodetic number;vertex strong geodetic number; connected strong geodetic number.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [1]. Two vertices u and v are said to be *adjacent* if uv is an edge of G. Two edges of G are said to be adjacent if they have a common vertex. The *distance*$d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G.

An $u-v$ path of length $d(u,v)$ is called an $u-v$*geodesic*.An $x-y$ path of length $d(x,y)$ is called geodesic. A vertex v is said to lie on a geodesic P if v is an internal vertex of P. The closed interval $I[x,y]$ consists of x, y and all vertices lying on some $x-y$ geodesic of G and for a non-empty set $S \subseteq V(G)$, $I[S] = \cup_{x,y \in S} I[x,y]$. A set $S \subseteq V(G)$ in a connected graph G is a geodetic set of G if $I[S] = V(G)$. The geodetic number of G, denoted by $g(G)$, is the minimum cardinality of a geodetic set of G.The geodetic concept were studied in [1, 3, 4]. Let x be a vertex of G and $S \subseteq V - \{x\}$. Then for each vertex$y \in S, x \neq y$. Let $\tilde{g}_x[y]$ be a selected fixed shortest x-y path. Then we set $\tilde{I}_x[S] = \{\tilde{g}_x(y): y \in S\}$ and let $V(\tilde{I}_x[S]) = \bigcup_{p \in \tilde{I}_x[S]} V(P)$. If $V(\tilde{I}_x[S]) = V$ for some $\tilde{I}_x[S]$then the set S is called a vertex strong geodetic set of G. The minimum cardinality of a vertex strong geodetic set of G is called the vertex strong geodetic number of G and is denoted by $sg_x(G)$.The following theorem is used in sequel.

Theorem 1.1[4] Each extreme vertex of a connected graph belong to every geodetic set of G.

2. The connected vertex strong geodetic number of a graph

Definition 2.1. Let x be a vertex of G and $S \subseteq V - \{x\}$. Then for each vertex$y \in S, x \neq y$. Let $\tilde{g}_x[y]$ be a selected fixed shortest x-y path. Then we set $\tilde{I}_x[S] = \{\tilde{g}_x(y): y \in S\}$ and let $V(\tilde{I}_x[S]) = \bigcup_{p \in \tilde{I}_x[S]} V(P)$. If $V(\tilde{I}_x[S]) = V$ for some $\tilde{I}_x[S]$then the set S is called a vertex strong geodetic set of G. A vertex strong geodetic set S of x of G is called a connected vertex strong geodetic set of G if G[S] is connected. The minimum cardinality of a connected vertex strong geodetic set of G is called the connected vertex strong geodetic number of G and is denoted by $csg_x(G)$.

Example 2.2. For the graph G given in Figure 2.1,csg_x-sets and $csg_x(G)$ for each vertex x is given in the following Table 2.1.

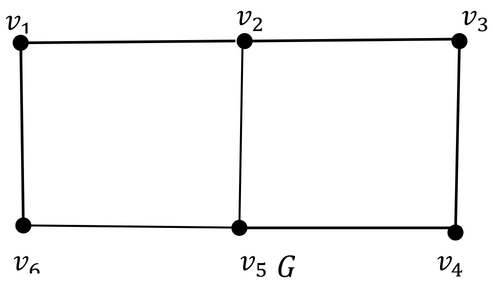

Figure 2.1

Table 2.1

Vertex	csg_x-sets	$csg_x(G)$
v_1	$\{v_3, v_4\}, \{v_4, v_5\}$	2
v_2	$\{v_4, v_5, v_6\}$	3
v_3	$\{v_1, v_6\}, \{v_5, v_6\}$	2
v_4	$\{v_1, v_6\}, \{v_1, v_2\}$	2
v_5	$\{v_1, v_2, v_3\}$	3
v_6	$\{v_3, v_4\}, \{v_2, v_3\}$	2

Observation 2.3. Let x be any vertex of a connected graph$G$.

(i) If $y \neq x$ be a simplicial vertex of G, then y belongs to every connected x-vertex strong geodetic set of G.

(ii) The eccentric vertices of x belong to every connected x-vertex strong geodetic set of G.

In the following we determine the connected vertex strong geodetic number of some standard graphs G for each vertex in G.

Theorem 2.4. For the path$G = P_n (n \geq 3)$,

$$csg_x(G) = \begin{cases} 1 \; if \; x \; is \; an \; end \; vertex \; of \; G \\ n \; if \; x \; is \; a \; cut \; vertex \; of \; G \end{cases}$$

Proof. Let $P_n$be $v_1, v_2, \ldots, v_n$.

If $x = v_1$, then $S = \{v_n\}$ is a csg_x-set of G so that $csg_x(G) = 1$. Similarly if $x = v_n$, then $csg_x(G) = 1$. Let x be a cut vertex of G. Then by Observation 2.3 (i)

$\{v_1, v_n\}$ is a subset of every csg_x-set of G. Let S be a csg_x-set of G. Since $G[S]$ is connected, it follows that $S = V(G)$ is the unique csg_x-set of G so that $csg_x(G) = n$.■

Theorem 2.5. For the cycle $G = C_n(n \geq 4)$, $csg_x(G) = 2$, for every $x \in G$.

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Without loss of generality let us assume that $x = v_1$.

Case (i) Let n be even. Let $n = 2k$ $(k \geq 2)$. Then v_{k+1} is the eccentric vertex of G. By Observation 2.3(ii) since $\{v_{k+1}\}$ is not a sg_x-set of G so that $csg_x(G) \geq 2$. Let $S = \{v_{k+1}, v_{k+2}\}$. Then S is a csg_x-set of G so that $csg_x(G) = 2$.

Case (ii) Let n be odd. Let $n = 2k + 1$ $(k \geq 2)$. Then $S = \{v_{k+1}, v_{k+2}\}$ is the eccentric vertices of G. By Observation 2.3 (ii) S is a subset of every csg_x-set of G and so $csg_x(G) \geq 2$. Since S is a sg_x-set of G and $G[S]$ is connected, S is a csg_x-set of G sothat $csg_x(G) = 2$.■

Theorem 2.6. For the complete graph $G = K_n(n \geq 4)$, $csg_x(G) = n - 1$, for every $x \in G$.

Proof. Let x be a vertex of G. Let $S = V(G) - \{x\}$. Since every vertex of G is an extreme vertex of G, it follows from Observation 2.3(i), S is the unique csg_x-set of G so that$csg_x(G) \geq n - 1$ for every vertex x in G.■

Theorem 2.7. For the fan graph $G = K_1 + P_{n-1}(n \geq 5)$.

$$csg_x(G) = \begin{cases} n-1 & if\ x \in V(K_1) \\ n-3 & if\ x\ is\ extreme\ vertex\ of\ P_{n-1} \\ n-2 & if\ x\ is\ internal\ vertex\ of\ P_{n-1} \end{cases}$$

Proof. Let $V(K_1) = y$ and $V(P_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$.

Case (i) Let$x = y$, Then $S = \{v_1, v_2, \ldots, v_n\}$ is a set of all eccentric vertices for x. By Observation 2.3 (ii) S is a subset of every csg_x-set of G and so $csg_x(G) \geq n - 1$. Since $G[S]$ is connected, S is a csg_x-set of G so that $csg_x(G) = n - 1$. Let $x \in V(P_{n-1})$. Let $x = v_1$. Then $S = \{v_3, v_4, \ldots, v_{n-1}\}$ are eccentric vertices of G. By Observation 2.3 (ii) S is a subset of every csg_x-set of G and so $csg_x(G) \geq n - 3$. Now S is a sg_x-set of G and $G[S]$ is connected. Therefore S is a csg_x-set of G so that $csg_x(G) = n - 3$. If $x = v_{n-1}$, by the similar way we can prove that $csg_x(G) = n - 3$. Let $x \in \{v_2, v_3, \ldots, v_{n-2}\}$. Without loss of generality let us assume that $x = v_2$. Then $\{v_1, v_{n-1}\}$ is set of extreme vertices of G. By Observation 2.3 (i) $\{v_1, v_{n-1}\}$ is a subset of every csg_x-set of $G.\{v_4, v_5, \ldots, v_{n-2}\}$ is the set of eccentric vertices of v_2. Then $\{v_4, v_5, \ldots, v_{n-2}\}$is a subset of every csg_x-set of G. Let $S' = \{v_1, v_4, v_5, \ldots, v_{n-2,} v_{n-1,}\}$. Then S' is a sg_x-set of G but $G[S']$ is not connected. Therefore $S' \cup \{y\}$ is a csg_x-set of G so that $csg_x(G) = n - 2$.■

Theorem 2.8. For the wheel graph $G = K_1 + C_{n-1}(n \geq 5)$.

$$csg_x(G) = \begin{cases} n-1 & if\ x \in v_1 \\ n-3 & if\ x \in V(C_{n-1}) \end{cases}$$

Proof. Let $V(K_1) = y$ and $V(C_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$.

Case(i) Let$x = y$, Then $S = \{v_1, v_2, \dots, v_{n-1}\}$ is a set of all eccentric vertices for x. By Observation 2.3 (ii) S is a subset of every csg_x-set of G and so $csg_x(G) \geq n-1$. Since $G[S]$ is connected, S is a csg_x-set of G so that $csg_x(G) = n-1$.

Case (ii) Let $x \in V(C_{n-1})$. Without loss of generality, let us assume that $x = v_1$. Then $S = \{v_3, v_4, \dots, v_{n-1}\}$ are eccentric vertices of G. By Observation 2.3 (ii) S is a subset of every csg_x-set of G and so $csg_x(G) \geq n-3$. Now S is a sg_x-set of G and $G[S]$ is connected. Therefore S is a csg_x-set of G so that $csg_x(G) = n-3$.■

Theorem 2.9. For the star graph $G = K_{1,n-1} (n \geq 3), csg_x(G) = n-1$ for every $x \in G$.

Proof. Let$y$ be the cut vertex of G and $\{v_1, v_2, \dots, v_{n-1}\}$ is a set of all eccentric vertices of G.Let$x = y$, Then $S = \{v_1, v_2, \dots, v_{n-1}\}$ is a set of all eccentric vertices for x. By Observation 2.3 (ii) S is a subset of every csg_x-set of G and so $csg_x(G) \geq n-1$. Since $G[S]$ is connected, S is a csg_x-set of G so that $csg_x(G) = n-1$. Let $x \in \{v_1, v_2, \dots, v_{n-1}\}$ Without loss of generality, let us assume that $x = v_1$. Then $S = \{v_2, v_3, \dots, v_{n-1}\}$ are set of eccentric vertices of v_1. By Observation 2.3 (ii) S is a subset of every sg_x-set of G and so $csg_x(G) \geq n-2$. Now S is a csg_x-set of G but $G[S]$ is not a csg_x-set of G and so $csg_x(G) \geq n-1$. Let $S' = S \cup \{x\}$. Then S' is a csg_x-set of G so that $csg_x(G) = n-1$.■

Theorem 2.10. For the Peterson graph $G, csg_x(G) = 6$ for every $x \in G$.

Proof.

Case (i) Let$x \in \{v_1, v_2, v_3, v_4, v_5\}$. Without loss of generality let us assume that $x = v_1$. Then $S = \{v_2, v_5, v_7, v_8, v_9, v_{10}\}$ is the set of all eccentric vertices for x. By Observation 2.3 (ii) S is a subset of every csg_x-set of G and so $csg_x(G) \geq 6$. Since S is a sg_x-set of G and $G[S]$ is connected, S is a csg_x-set of G so that $csg_x(G) = 6$.

Case (ii) Let$x \in \{v_6, v_7, v_8, v_9, v_{10}\}$. Without loss of generality let us assume that $x = v_6$. Then $S = \{v_2, v_3, v_4, v_5, v_8, v_9\}$ is the set of all eccentric vertices for x. By Observation 2.3 (ii) S is a subset of every csg_x-set of G and so $csg_x(G) \geq 6$. Since S is a sg_x-set of G and $G[S]$ is connected, S is a csg_x-set of G so that $csg_x(G) = 6$.■

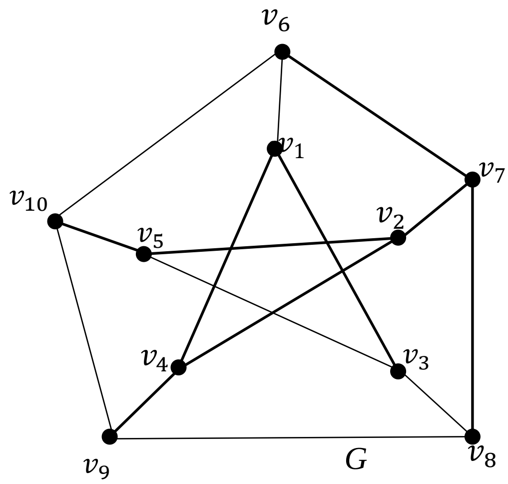

Figure 2.2

Theorem 2.11. Let G be a connected graph. Then $1 \leq sg_x(G) \leq csg_x(G) \leq n$ for every vertex x in G.

Proof. Let x be a vertex of G. Since every sg_x-set of G needs at least one vertex $sg_x(G) \geq 1$. Since every connected strong vertex geodetic set of G is a strong vertex geodetic set of $G, sg_x(G) \leq csg_x(G)$. Since $V(G)$ is a connected strong vertex geodetic set of $G, csg_x(G) \leq n$. Therefore $1 \leq sg_x(G) \leq csg_x(G) \leq n$.■

Theorem 2.12. Let G be a connected graph. Then $csg_x(G) = 1$ if and only if x is an end vertex of $P_n (n \geq 2)$.

Proof. Let x be an end vertex of P_n. Then by Theorem 2.4, $csg_x(G) = 1$. Conversely let $csg_x(G) = 1$. Let $S = \{y\}$ be the csg_x-set of x. We prove that x is an end vertex of P_n. On the contrary suppose that x is not an end vertex of P_n. Then there are at least two $x - y$ geodesics, which is a contradiction to S a csg_x-set of G. Therefore x is an end vertex of P_n.■

Theorem 2.13. Let G be a connected graph and $x \in G$. If x is a universal vertex of G. Then $csg_x(G) = n - 1$.

Proof. Let x be a universal vertex of G. Then $V(G) - \{x\}$ is set of all eccentric vertices for x. By Observation 2.3 (ii), S is a subset of every csg_x-set of G and so $csg_x(G) \geq n - 1$. Since $G[S]$ is connected, S is a csg_x-set of G so that $csg_x(G) = n - 1$.■

Theorem 2.14. Let G be a connected graph and $x \in G$. If x is a cut vertex and universal vertex of G. Then$csg_x(G) = n$.

Proof. Since x is a universal vertex of G, then $V(G) - \{x\}$ is set of all eccentric vertices for x. By Observation 2.3 (ii), S is a subset of every csg_x-set of G and so

$csg_x(G) \geq n - 1$. Since $G[S]$ is not connected, S is not a csg_x-set of G. Therefore $S = V(G)$ is the unique csg_x-set of G. Hence $csg_x(G) = n$.■

Theorem 2.15. For every pair of integers$a$ and b with $2 \leq a \leq b$, there exists a connected graph G such that $sg_x(G) = a$ and $csg_x(G) = b$ for some vertex x in G.

Proof. For $a = b$, let $G = K_{a+1}$. Then by Theorem 2.11$sg_x(G) = csg_x(G) = a$ for every vertex x in G. For $b = a + 1$,let $G = K_{1,a}$. Let x be a universal vertex of G. Then by Theorem 2.14,$sg_x(G) = a$ and $csg_x(G) = a + 1$. So, let $b \geq a + 2$. Let $P_0: u_0, u_1, u_2, \dots, u_{b-a}, u_{b-a+1}$be a path of order $b - a + 2$. Let G be the graph obtained from P by adding the new vertices $z_1, z_2, \dots, z_{a-1}$ and introducing the edges $z_i u$ $(1 \leq i \leq b - a + 1)$. The graph $G$is shown in Figure 2.3. Let $x = u_{b-a+1.}$

First we prove that $sg_x(G) = a$.Let$S = \{u_0, z_1, z_2, \dots, z_{a-1}, u_{b-a+1}\}$be the end vertices of G. By Observation 2.3(i), $S_1 = S - \{u_{b-a+1}\}$ is a subset of every sg_x-set of G and so $sg_x(G) \geq a$. Since S_1 is a sg_x-set of $G, sg_x(G) = a$.

Next we prove that $csg_x(G)$= b.By Observation, S_1 is a subset of everycsg_x-set of G. Since $G[S_1]$ is not connected S_1 is not a csg_x-set of G. let $S_2 = S_1 \cup \{u_1, u_2, \dots, u_{b-a}\}$. Then S_2 is a csg_x-set of G and$G[S_2]$ is connected. Therefore S_2 is a csg_x-set of G so that , $csg_x(G) = b$.■

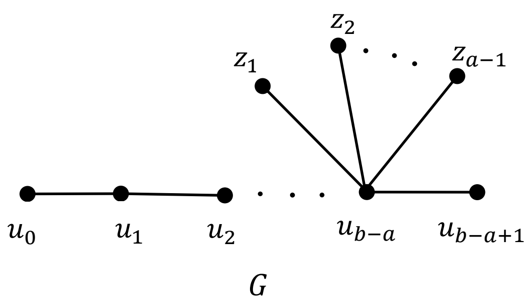

Figure 2.3

3. Conclusions

In this article we explore the concept of the forcing strong geodetic number of a graph. We extend this concept to some other distance related parameters in graphs.

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