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Modular Coloring and Switching in Some Planar Graphs

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Abstract

For a connected graph G, let c: V (G) $\rightarrow \mathbb{Z}_k$ ($k \geq 2$) be a vertex coloring of G. The color sum σ(v) of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in N (v) that is (v) = $\sum_{u \in N(v)} c(u)$ (mod k). The coloring c is called a modular k-coloring of G if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs of adjacent vertices x, y ∈ G. The modular chromatic number or simply the mc-number of G is the minimum k for which G has a modular k-coloring. A switching graph is an ordinary graph with switches. For many problems, switching graphs are a remarkable straight forward and natural model, but they have hardly been studied. A vertex switching G_v of a graph G is obtained by taking a vertex V of G, removing the entire edges incident with V and adding edges joining V to every vertex which are not adjacent to V in G. In this paper we determine the modular chromatic number of Wheel graph, Friendship graph and Gear graph after switching on certain vertices. Here, we first define switching of graphs. Next, we investigating several problems on finding the mc(G) after switching of graphs and provide their characterization in terms of complexity.

Keywords. Modular coloring, Modular chromatic number, Switching, Wheel Graph, Friendship graph, Gear graph**.**

MSC AMS Classification 2020: 05C15[3]

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1. Introduction

We are encouraged by the modular colorings and the modular chromatic number of different graphs, where the chromatic number is defined as the color sum of all the neighboring vertices in $\mathbb{Z}_k$. At this point of view, to the curiosity for minimizing the modular chromatic number, determined to switching in certain vertices in some graphs.

For a vertex v of a graph G, let N (v) denote the neighborhood of v (the set of adjacent vertices to vertex v). For a graph G without isolated vertices, let c: V (G) → $\mathbb{Z}_k$ ($k \geq 2$) be a vertex coloring of G where adjacent vertices may be colored the same. The color sum σ(v) of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in N(v) ,that is $\sigma(v) = \sum_{u \epsilon N(v)} c(u)$[1, 2, 3]. The coloring c is called a modular sum k-coloring or simply a modular k-coloring of G, if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs x, y of adjacent vertices of G. A coloring c is called modular coloring if c is a modular k-coloring for some integar $k \geq 2$.The modular chromatic number mc(G) is the minimum k for which G has a modular k- coloring.This concept was introduced by Okamoto, Salehi and Zhang [4, 5, 6, 8].
In order to distinguish the vertices of a connected graph and to differntiate the adjacent vertices of a graph with the minimum number of colors, the concept of modular coloring was put forward by Okamoto, Salehi and Zhang [6].

A graph H is the switching of a graph G with respect to the vertex v of G if V (G) = V (H) and E(H) = (E(G)\{ uv : u ∈ $N_G(v)$ }) ∪ { uv : u ∈$N'_G(v)$ }[7,9]. The switching of G with respect to v is denoted Gs (v). The operation of creating Gs (v) is called switching on v in G. In other words, switching on a vertex v of a graph has the effect of removing all edges incident with the vertex v and joining the vertex v to all vertices to which it was formerly non-adjacent. Here, we first define switching of graphs. Next, we investigating several problems on finding the mc (G) after switching of graphs and provide their characterization in terms of complexity. In this paper we find the modular chromatic number of wheel graph, friendship graph and gear graph after switching on certain vertices at different levels.

2. Modular coloring of wheel graph after switching

The switching of a vertex in a wheel having n vertices is denoted by W_s (n).
In Wheel switching is not possible in W (3) since it is a complete graph. Switching in a wheel is obtained in two ways. They are
1)The switching at the vertex u ∈ ℓ_0.
2)The switching at the vertex v_i ∈ ℓ_1.be the vertices
Let u ∈ ℓ_0 be the central vertex and $v_1, v_2, \ldots, v_n$ ∈ $\ell_1$be the vertices which are adjacent to u ∈ ℓ_0. The switching at u ∈ ℓ_0 makes the graph W(n) is a cycle having n vertices with a central vertex u ∈ ℓ_0.

Theorem 2.1
The modular coloring of a graph obtained after the switching of a vertex $v_i \in \ell_1$ is $W_s(n) = 3$ for n = 4k, 4k+1, [4k+2; k > 1]; $W_s(6) = 4$; $W_s(n) = 4$ for n = 4k+3, k ≥ 1.
Proof:
For a wheel W (n), let the vertex $u \in \ell_0$, $v_i \in \ell_1$ for I = 1, 2, …., n be the vertices.
Case (i) mc [$W_s(4)$] = 3.

Let $v_1, v_2, v_3, v_4 \in \ell_1$ be the vertices at level ℓ=1.Switching is taken for $v_1 \in \ell_1$.
Consider the modular coloring c(v):v[$W_s(4)$]→ $\mathbb{Z}_3$ defined by

$$c(v)=\begin{cases} 0 \text{ for } v_1 \in \ell_1, u \in \ell_0 \\ 1 \quad \text{otherwise} \end{cases}$$

then $\sigma(v)=\begin{cases} 0\ for\ u \in \ell_0 \\ 2\ for\ v_3 \in \ell_1 \\ 1\ otherwise \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in $W_s(4)$.
∴ mc [$w_s(4)$]=3. Hence the proof.

Case (ii) mc [$W_s(5)$] = 3.
Let $v_1, v_2, v_3, v_4, v_5 \in \ell_1$ be the vertices at level ℓ=1.Switching is taken for $v_1 \in \ell_1$.
Consider the modular coloring c (v):v[$W_s(5)$]→ $\mathbb{Z}_3$ defined by $c(v)=\begin{cases} 1 \text{ for } v_3 \in \ell_1 \\ 2 \text{ for } v_4 \in \ell_1 \\ 0 \quad \text{otherwise} \end{cases}$

then $\sigma(v) =\begin{cases} 0 \text{ for } u \in \ell_0, v_1 \in \ell_1 \\ 1 \quad \text{for } v_{2,} v_4 \in \ell_1 \\ 2 \quad \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ ∀x,y of adjacent vertices in $W_s(5)$.
∴ mc [$w_s(5)$]=3.Hence the proof.

Case (iii)mc[$W_s(6)$] = 4. Let $v_1,v_2,v_3,v_4,v_5,v_6 \in \ell_1$ be the vertices at level ℓ=1.Switching is taken for $v_1 \in \ell_1$ Consider the modular coloring c(v):v[$W_s(6)$]→ $\mathbb{Z}_3$ defined by c(v) = $\begin{cases} 1 \text{ for } v_3 \in \ell_1 \\ 2 \text{ for } v_4 \in \ell_1 \\ 0 \quad \text{otherwise} \end{cases}$

Then $\sigma(v) = \begin{cases} 3 \text{ for } u \in \ell_0, v_1 \in \ell_1 \\ 1 \quad \text{for } v_{2,} v_4 \in \ell_1 \\ 2 \quad \text{for } v_{3,,} v_5 \in \ell_1 \\ 0 \quad \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in $W_s(6)$.
∴ mc [$w_s(6)$] = 3. Hence the proof.

Case (iv) mc [$W_s(4k+3)$] = 4.
Let $u \in \ell_0$ be the central vertex. Let $v_1,v_2,\ldots v_{i-1},v_i,v_{i+1},\ldots,v_{4k+3} \in \ell_1$ be the vertices at level ℓ=1.Switching is taken for $v_i \in \ell_1$. After switching v_i is adjacent to the vertices $v_{i+2},v_{i+3},\ldots v_{4k+3},v_1,v_2,\ldots,v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1}. The 4k vertices which are adjacent to v_i is renamed as $R_1, R_2, \ldots, R_{4k}$.

Subcase (i) mc [W_s (4k+3)] = 4 for k = 1 + 4j, j = 0, 1, 2, …. [Eg: Ws (7), Ws (23), Ws (39), ….

Consider the modular coloring c(v):v[W_s(4k+3)]→ $\mathbb{Z}_4$ defined by

$$c(v)=\begin{cases} 2 & \text{if } R_{4k} \in \ell_1 \\ 1 \text{ if } R_{1+4j} \in \ell_1, j = 0,1,2 \ldots (k-1) & \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v)=\begin{cases} 3 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 2 & \text{for } v_{i-1}, R_{4k-1} \in \ell_1 \\ 1 \text{ for } v_{i+1,}, R_{2j} \in \ell_1, \text{for } j = 1,2, \ldots (2k-1) & \\ 0 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in W_s(4k+3). ∴ mc[w_s(4k+3)]=4 for k=1+4j ,j=0,1,2,….Hence the proof.

Eg:

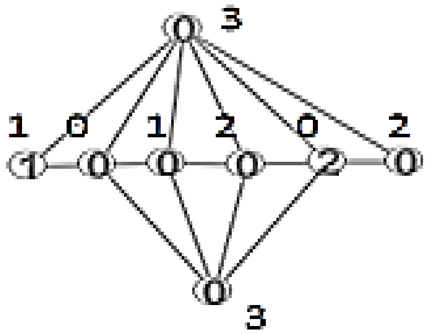

Figure1. Switching with modular coloring in Ws(7)

Subcase(ii) mc[W_s(4k+3)]=4 for k=2+4j ,j=0,1,2,….[Ws(11),Ws(27), Ws(43),…]

Consider the modular coloring c(v):v[W_s(4k+3)]→ $\mathbb{Z}_4$ defined by

$$c(v)=\begin{cases} 2 & \text{if } R_{4k} \in \ell_1 \\ 1 \text{ if } u \in \ell_0, R_{1+4j} \in \ell_1, j = 0,1,2 \ldots (k-1) & \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v)=\begin{cases} 0 & \text{for } u \in \ell_0, v_i \in \ell_1 \\ 3 & \text{for } R_{4k-1} \in \ell_1 \\ 2 \text{ for } v_{i+1,}, v_{i-1}, R_{2j} \in \ell_1, \text{for } j = 1,2, \ldots (2k-1) & \\ 1 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in W_s(4k+3). ∴ mc[w_s(4k+3)]=4 for k=2+4j ,j=0,1,2,…. Hence the proof.

Eg:

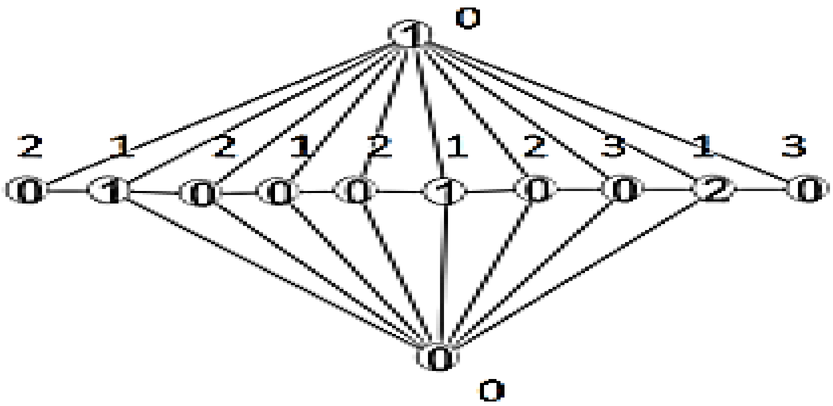

Figure 2. Switching with modular coloring in Ws(11)

Subcase(iii) mc[W$_s$(4k+3)]=4 for k=3+4j ,j=0,1,2,….[Eg: Ws(15), Ws(31),Ws(47),…]

Consider the modular coloring c(v):v[W$_s$(4k+3)]→ $\mathbb{Z}_4$ defined by

$$c(v)=\begin{cases} 2 & \text{if } u \in \ell_0,\ R_{4k} \in \ell_1 \\ 1 \text{ if } R_{1+4j} \in \ell_1, j = 0,1,2 \ldots (k-1) & \\ 0 & \text{elsewhere} \end{cases}$$

then $$\sigma(v)=\begin{cases} 0 & \text{for } v_{i-1}, R_{4k-1} \in \ell_1 \\ 1 & \text{for } u \in \ell_0,\ v_i \in \ell_1 \\ 3 \text{ for } v_{i+1,,} R_{2j} \in \ell_1, \text{for } j = 1,2, \ldots (2k-1) & \\ 2 & \text{otherwise} \end{cases}$$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in W$_s$(4k+3). ∴ mc[w$_s$(4k+3)]=4 for k=3+4j ,j=0,1,2,….Hence the proof.

Eg:

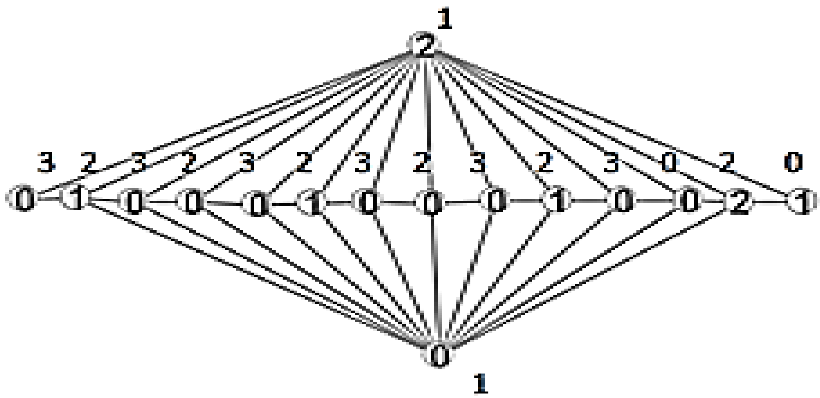

Figure 3. Switching with modular coloring in Ws(15)

Subcase(iv) mc[W$_s$(4k+3)]=4 for k=4+4j ,j=0,1,2,….[Ws(19),Ws(35),Ws(51),…]

Consider the modular coloring c(v):v[W_s(4k+3)]→ $\mathbb{Z}_4$ defined by

$$c(v)=\begin{cases} 3 & \text{if } u \in \ell_0 \\ 1 \text{ if } R_{1+4j} \in \ell_1, j=0,1,2\ldots(k-1) & \\ 2 & \text{if } R_{4k} \in \ell_1 \\ 0 & \text{elsewhere} \end{cases}$$

then $$\sigma(v)=\begin{cases} 1 & \text{for } v_{i-1}, R_{4k-1} \in \ell_1 \\ 2 & \text{for } u \in \ell_0,\ v_i \in \ell_1 \\ 0 \text{ for } v_{i+1,,} R_{2j} \in \ell_1, \text{for } j=1,2,\ldots(2k-1) & \\ 3 & \text{otherwise} \end{cases}$$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in W_s(4k+3). ∴ mc[w_s(4k+3)]=4 for k=4+4j ,j=0,1,2,….Hence the proof.

Eg:

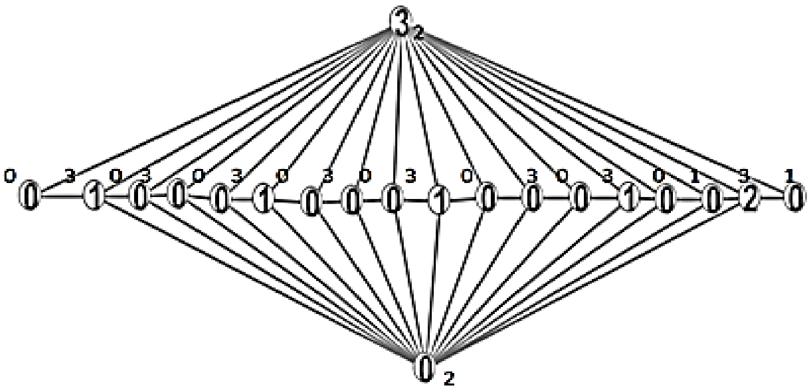

Figure 4. Switching with modular coloring in Ws(19)

Case(v)mc[W_s(4k)] = mc[W_s(4k+1)] = mc[W_s(4k+2)]=3.

Subcase(i) mc[W_s(4k)] = 3 for k=2+3j,j=0,1,2….[Ws(8),Ws(20),Ws(32),…]

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \ldots v_{4k}, v_1, v_2, \ldots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1}. Let the 4k-3 vertices which are adjacent to v_i is renamed as $R_1, R_2, \ldots, R_{4k-3}$ respectively. Consider the modular coloring c(v):v[W_s(4k)]→ $\mathbb{Z}_3$ defined by $c(v)=\begin{cases} 1 \text{ if } R_{1+4j} \in \ell_1, j=0,1,2\ldots(k-1) & \\ 0 & \text{elsewhere} \end{cases}$

then $$\sigma(v)=\begin{cases} 2 & \text{for } u \in \ell_0,\ v_i \in \ell_1 \\ 1 \text{ for } v_{i+1,,} v_{i-1}, R_{2j} \in \ell_1, \text{for } j=1,2,\ldots,2(k-1) & \\ 0 & \text{otherwise} \end{cases}$$

Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in W_s(4k). ∴ mc[w_s(4k)]=3 for k=2+3j ,j=0,1,2,….Hence the proof.

Eg:

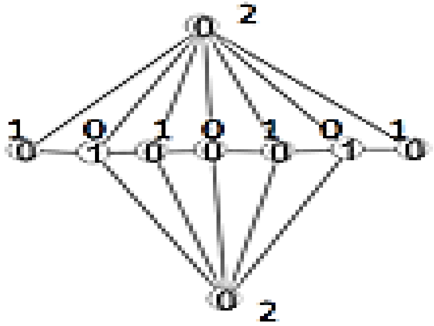

Figure 5. Switching with modular coloring in Ws(8)

Subcase(ii) mc[W_s(4k)] = 3 for k=3+3j,j=0,1,2….[Ws(12),Ws(24),Ws(36),..]

After switching v_i is adjacent to the vertices $v_{i+2},v_{i+3},\dots v_{4k},v_1,v_2,\dots,v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1}.
Let the 4k-3 vertices which are adjacent to v_i is renamed as $R_1, R_2, \dots, R_{4k-3}$ respectively.
Consider the modular coloring c(v):v[W_s(4k)]→ $\mathbb{Z}_3$ defined by

$$c(v)=\begin{cases} 1 \text{ if } u \in \ell_0,\ R_{1+4j} \in \ell_1, j = 0,1,2 \dots (k-1) \\ 0 \quad \text{elsewhere} \end{cases}$$

then $$\sigma(v)=\begin{cases} 0 & \text{for } u \in \ell_0,\ v_i \in \ell_1 \\ 1 & R_{1+2j} \in \ell_1, \text{for } j = 0,1,2,\dots,(k+1) \\ 2 & \text{otherwise} \end{cases}$$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in W_s(4k). ∴ mc[w_s(4k)]=3 for k=3+3j ,j=0,1,2,….Hence the proof.
Eg:

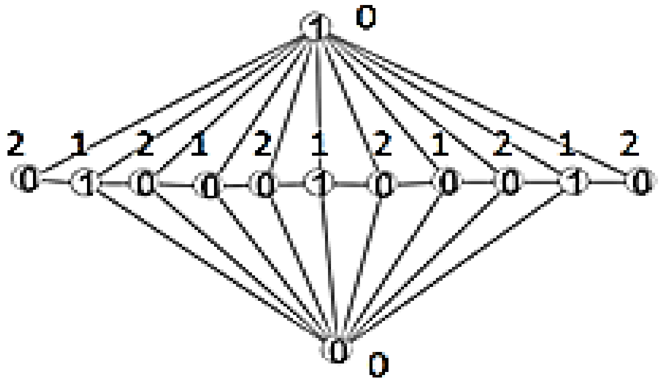

Figure 6. Switching with modular coloring in Ws(12)

Subcase(iii) mc[W_s(4k)] = 3 for k=4+3j,j=0,1,2….[Ws(16),Ws(28),Ws(40),…]
After switching v_i is adjacent to the vertices $v_{i+2},v_{i+3},\dots v_{4k},v_1,v_2,\dots,v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1}.
Let the 4k-3 vertices which are adjacent to v_i is renamed as $R_1, R_2, \dots, R_{4k-3}$ respectively.

Consider the modular coloring c(v):v[$W_s(4k)$]→ $\mathbb{Z}_3$ defined by

$$c(v)=\begin{cases} 2 & \text{if } u \in \ell_0 \\ 1 \text{ for} R_{1+4j} \in \ell_1, j = 0,1,2 \ldots (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v)=\begin{cases} 1 & \text{for } \; u \in \ell_0, \; v_i \in \ell_1 \\ 2 \; R_{1+2j} \in \ell_1, \text{for } j = 0,1,2,\ldots,2(k-1) \\ 0 & \text{otherwise} \end{cases}$

Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in $W_s(4k)$.

∴ mc[$w_s(4k)$]=3 for k=4+3j ,j=0,1,2,….

Hence the proof.

Eg:

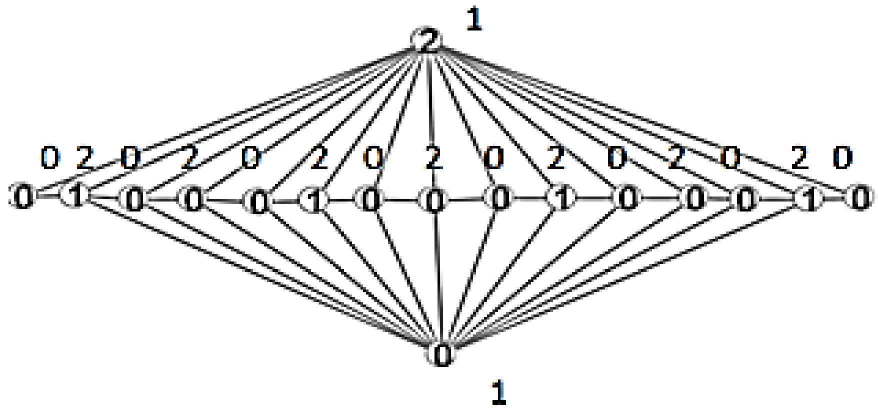

Figure7. Switching with modular coloring in Ws(16)

Subcase(iv) mc[$W_s(4k+1)$] = 3 for k=2+3j,j=0,1,2….[Ws(9),Ws(21),Ws(33),…]

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \ldots v_{4k+1}, v_1, v_2, \ldots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1}.

Let the 4k-2 vertices which are adjacent to v_i is renamed as $R_1, R_2 \ldots R_{4k-2}$ respectively.

Consider the modular coloring c(v): v[$W_s(4k+1)$]→ $\mathbb{Z}_3$ defined by c(v) =

$$\begin{cases} 1 \text{ for} R_{1+4j} \in \ell_1, j = 0,1,2 \ldots (k-1) \\ 0 & \text{elsewhere} \end{cases}$$

Then $\sigma(v) = \begin{cases} 2 & \text{for } \; u \in \ell_0, \; v_i \in \ell_1 \\ 1 \text{ for } v_{i+1}, R_{2j} \in \ell_1, \text{for } j = 1,2,\ldots,(2k-1) \\ 0 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ ∀ x, y of adjacent vertices in W_s (4k+1). ∴ mc [w_s (4k+1)] = 3 for k = 2+3j, j = 0, 1, 2, ….

Hence the proof.

Eg:

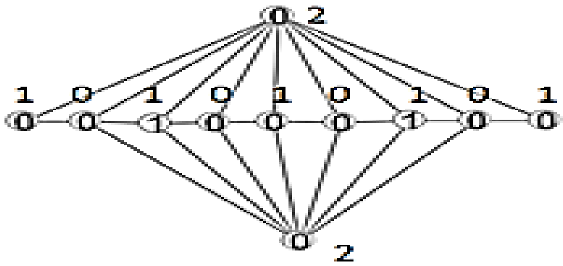

Figure 8. Switching with modular coloring in Ws(9)

Subcase (v) mc[W_s (4k+1)] = 3 for k = 3+3j, j = 0, 1, 2…. [Ws (13), Ws (25), Ws (37), …]

After switching v_i is adjacent to the vertices v_{i+2}, v_{i+3}, …v_{4k+1}, v_1, v_2,…,v_{i-2} respectively and not adjacent to the vertices v_{i-1} and v_{i+1}.
Let the 4k-2 vertices which are adjacent to v_i is renamed as R_1, R_2, ….., R_{4k-2} respectively.
Consider the modular coloring c(v):v[W_s(4k+1)]→ $\mathbb{Z}_3$ defined by

$$c(v)=\begin{cases} 1 \text{ for } u \in \ell_0 \,;\, R_{1+4j} \in \ell_1, j = 0,1,2 \ldots (k-1) \\ 0 \qquad\qquad \text{elsewhere} \end{cases}$$

Then $\sigma(v) = \begin{cases} 0 & \text{for } \; u \in \ell_0,\; v_i \in \ell_1 \\ 2 \text{ for } v_{i+1}, R_{2j} \in \ell_1, \text{for } j = 1,2,\ldots,(2k-1) & \\ 1 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in W_s(4k+1). ∴ mc[w_s(4k+1)] = 3 for k = 3+3j, j = 0, 1, 2, ….
Hence the proof.
Eg:

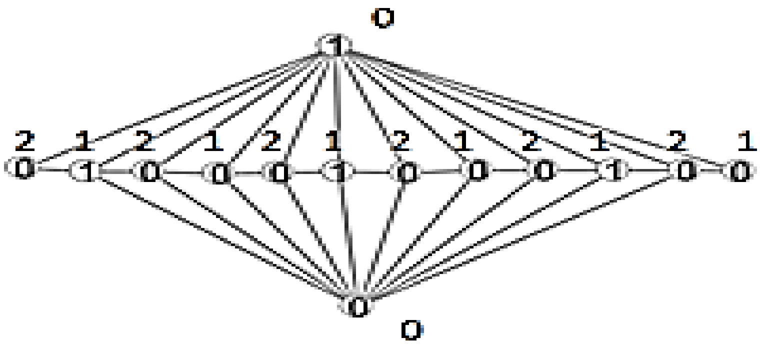

Figure 9. Switching with modular coloring in Ws(13)

Subcase (vi) mc [W_s(4k+1)] = 3 for k = 4+3j, j = 0, 1, 2…. [Ws (17), Ws (29), Ws (41), …]

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \ldots v_{4k+1}, v_1, v_2, \ldots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1}. Let the 4k-2 vertices which are adjacent to v_i is renamed as $R_1, R_2, \ldots, R_{4k-2}$ respectively.

Consider the modular coloring $c(v): v[W_s(4k+1)] \rightarrow \mathbb{Z}_3$ defined by

$$c(v)=\begin{cases} 2 & \text{for } u \in \ell_0 \\ 1 \text{ for } R_{1+4j} \in \ell_1, j = 0,1,2 \ldots (k-1) & \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v)=\begin{cases} 1 & \text{for } u \in \ell_0,\ v_i \in \ell_1 \\ 0 \text{ for } v_{i+1}, R_{2j} \in \ell_1, \text{for } j = 1,2,\ldots,(2k-1) & \\ 2 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ $\forall$ x,y of adjacent vertices in $W_s(4k+1)$. $\therefore$ $mc[w_s(4k+1)]=3$ for $k=4+3j$, $j=0,1,2,\ldots$.Hence the proof.

Eg:

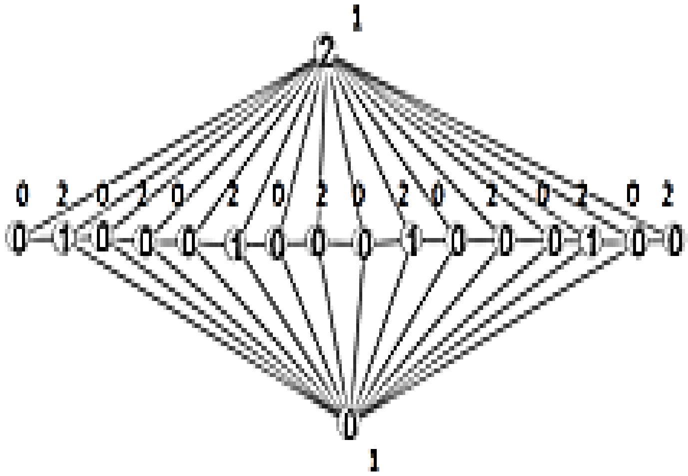

Figure 10. Switching with modular coloring in Ws(17)

Subcase(vii) $mc[W_s(4k+2)] = 3$ for $k=2+3j, j=0,1,2\ldots$.[Ws(10),Ws(22),Ws(34),…]

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \ldots v_{4k+2}, v_1, v_2, \ldots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1}.

Let the 4k-2 vertices which are adjacent to v_i is renamed as $R_1, R_2, \ldots, R_{4k-1}$ respectively.

Consider the modular coloring $c(v): v[W_s(4k+2)] \rightarrow \mathbb{Z}_3$ defined by

$$c(v)=\begin{cases} 1 \text{ for } R_{2+4j} \in \ell_1, j = 0,1,2 \ldots (k-1) & \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v)=\begin{cases} 2 & \text{for } u \in \ell_0,\ v_i \in \ell_1 \\ 1 \text{ for } R_{1+2j} \in \ell_1, \text{for } j = 0,1,2,\ldots,(2k-1) & \\ 0 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ $\forall$ x,y of adjacent vertices in $W_s(4k+2)$. $\therefore$ $mc[w_s(4k+2)] = 3$ for $k = 2+3j$, $j = 0, 1, 2, \ldots$. Hence the proof.

Eg:

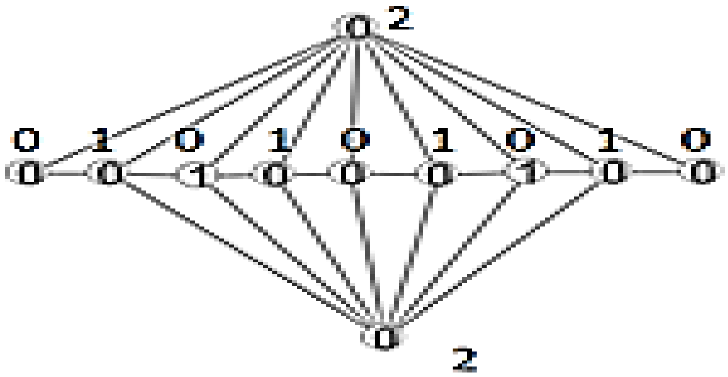

Figure 11. Switching with modular coloring in Ws(10)

Subcase(viii) mc[W_s(4k+2)] = 3 for k=3+3j,j=0,1,2….[Ws(14),Ws(26),Ws(38),…]

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \ldots v_{4k+2}, v_1, v_2, \ldots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1}. Let the 4k-2 vertices which are adjacent to v_i is renamed as $R_1, R_2, \ldots, R_{4k-1}$ respectively.

Consider the modular coloring c(v):v[W_s(4k+2)]→ $\mathbb{Z}_3$ defined by

$$c(v)=\begin{cases} 1 \text{ for } u \in \ell_0, R_{2+4j} \in \ell_1, j = 0,1,2 \ldots (k-1) \\ 0 \qquad\qquad \text{elsewhere} \end{cases}$$

then $$\sigma(v)=\begin{cases} 0 \qquad \text{for} \quad u \in \ell_0, \ v_i \in \ell_1 \\ 2 \text{ for } R_{1+2j} \in \ell_1, \text{for } j = 0,1,2,\ldots, (2k-1) \\ 1 \qquad \text{otherwise} \end{cases}$$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in W_s(4k+2). ∴ mc[w_s(4k+2)]=3 for k=3+3j ,j=0,1,2,….Hence the proof.

Eg:

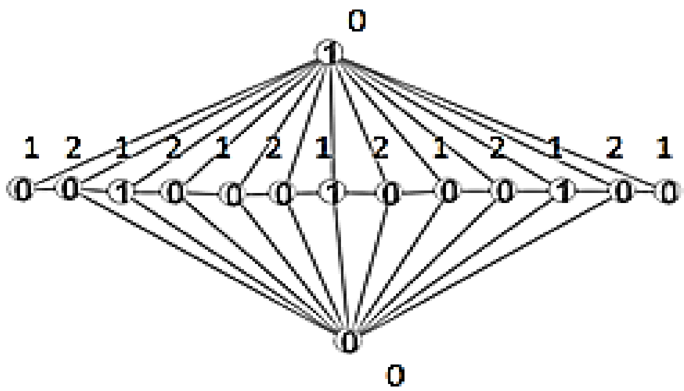

Figure 12. Switching with modular coloring in Ws(14)

Subcase(ix)mc [W_s(4k+2)] = 3 for k=4+3j,j=0,1,2….[Ws(18),Ws(30),Ws(42),…]

After switching v_i is adjacent to the vertices $v_{i+2}, v_{i+3}, \ldots v_{4k+2}, v_1, v_2, \ldots, v_{i-2}$ respectively and not adjacent to the vertices v_{i-1} and v_{i+1}. Let the 4k-2 vertices which are adjacent to v_i is renamed as $R_1, R_2, \ldots, R_{4k-1}$ respectively.

Consider the modular coloring $c(v): v[W_s(4k+2)] \rightarrow \mathbb{Z}_3$ defined by

$$c(v)=\begin{cases} 2 & \text{for } u \in \ell_0 \\ 1 \; for \; R_{2+4j} \in \ell_1, j = 0,1,2 \ldots (k-1) & \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v)=\begin{cases} 1 & \text{for } u \in \ell_0, \; v_i \in \ell_1 \\ 0 \text{ for } R_{1+2j} \in \ell_1, \text{for } j = 0,1,2, \ldots, (2k-1) & \\ 2 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ $\forall$ x,y of adjacent vertices in $W_s(4k+2)$. $\therefore$ $mc[w_s(4k+2)]=3$ for k=4+3j ,j=0,1,2,….Hence the proof.

Eg:

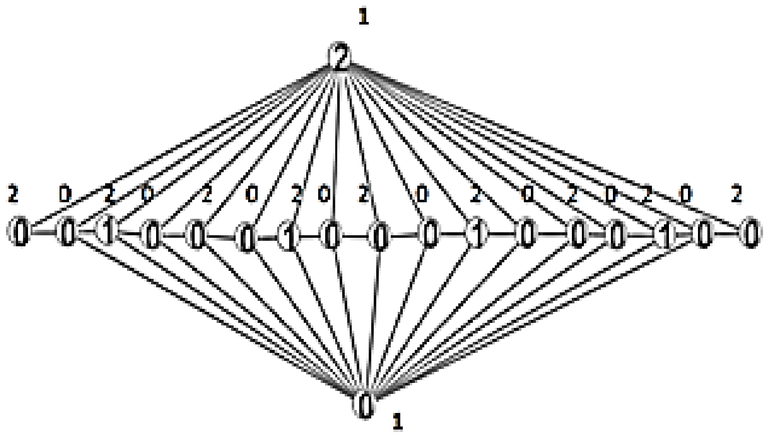

Figure 13. Switching with modular coloring in Ws(18)

3. Modular colorings after switching on Friendship graph

Let $u \in \ell_0$ be the center $v_1, v_2, v_3, \ldots v_{2n}$ be the vertices in ℓ_1 where each of 2 consecutive vertices forms an edge for the respective cycles since a friendship graph is constructed by joining n copies of the cycle C_3 with a common vertex. The vertices in ℓ_1 is taken in the clockwise direction. Modular coloring after switching of a friendship graph with n vertices is denoted by $mc[FS_s(n)]$.Here switching can be taken only for $v_i \in \ell_1$ for any i=1,2,… 2n.We cannot form a switching with $u \in \ell_0$ since it is adjacent to all vertices of ℓ_1.

Theorem 3.1.

The modular coloring of a friendship graph after switching a vertex in ℓ_1 then (i)$mc[FS_s(2)]=3$. (ii) $mc[FS_s(n)]=4; n \geq 3$.

Proof:

Case (i) $mc[FS_s(2)] = 3$.

Switching is taken for $v_1 \in \ell_1$. Therefore v_1 is adjacent to v_3 and v_4 after switching in $FS_s(2)$.

Consider a modular coloring $c(v)$:$v[FS_s(2)] \to \mathbb{Z}_3$ defined by $C(v)=\begin{cases} 2 & \text{for } u \in \ell_0 \\ 1 & \text{for } v_4 \in \ell_1 \\ 0 & \text{elsewhere} \end{cases}$

then $\sigma(v)=\begin{cases} 1 & \text{for } u \in \ell_0, v_1 \in \ell_1 \\ 2 & \text{for } v_4, v_2 \in \ell_1 \\ 0 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ ∀x,y of adjacent vertices in $FS_s(2)$

∴$mc[FS_s(2)]=3$. Hence the proof.

Case(ii) $mc[FS_s(n)]=4; n\geq 3$.

Subcase (i) $mc[FS_s(n)]=4; n\geq 3$ for n is odd.

Switching is taken for $v_1 \in \ell_1$. Therefore v_1 is adjacent to $v_3, v_4, \ldots, v_{2n}$ after switching in $FS_s(n)$.

Consider a modular coloring $c(v)$:$v[FS_s(n)] \to \mathbb{Z}_4$ defined by

$$C(v)=\begin{cases} 3 & \text{for } u \in \ell_0 \\ 2 & \text{for } v_{2j} \in \ell_1 ;\ j = 1,2,\ldots,n \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v)=\begin{cases} 2 & \text{for } u \in \ell_0 \\ 3 & v_{2j} \in \ell_1; j = 1,2,\ldots,n \\ 0 & \text{for } v_1 \in \ell_1 \\ 1 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in $FS_s(n)$ ∴ $mc[FS_s(n)]=4$ for $n\geq 3$ for n is odd. Hence the proof.

Eg:

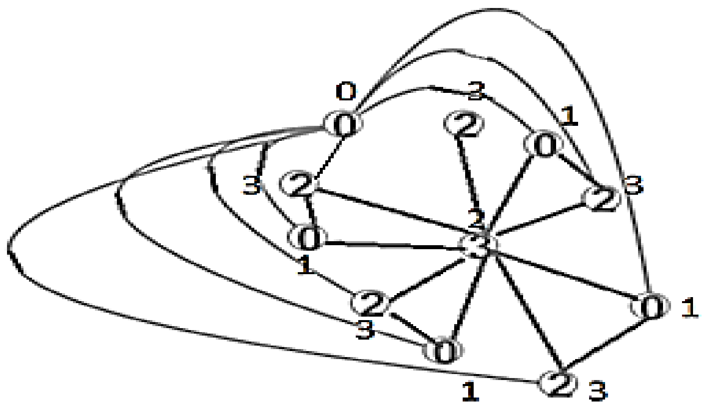

Figure 14. Switching with modular coloring in FSs(5)

Subcase (ii).mc[FS$_s$(n)]=4;n≥3 for n is even.
Switching is taken for $v_1 \in \ell_1$.Therefore v_1 is adjacent to v_3 ,v_4 ,….v_{2n} after switching in FS$_s$(n).
Consider a modular coloring c(v):v[FS$_s$(n)] → $\mathbb{Z}_4$ defined by

$$C(v)=\begin{cases} 3 & \text{for } u \in \ell_0 \\ 2 & \text{for } v_{2j} \in \ell_1 \,;\, j = 1,2,\ldots,n \\ 0 & \text{elsewhere} \end{cases}$$

then $\sigma(v)=\begin{cases} 0 & \text{for } u \in \ell_0 \\ 3 & v_{2j} \in \ell_1; j = 1,2,\ldots.,n \\ 2 & \text{for } v_1 \in \ell_1 \\ 1 & \text{otherwise} \end{cases}$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in FS$_s$(n) ∴ mc[FS$_s$(n]=4 for n≥3 for n is even. Hence the proof.

Eg:

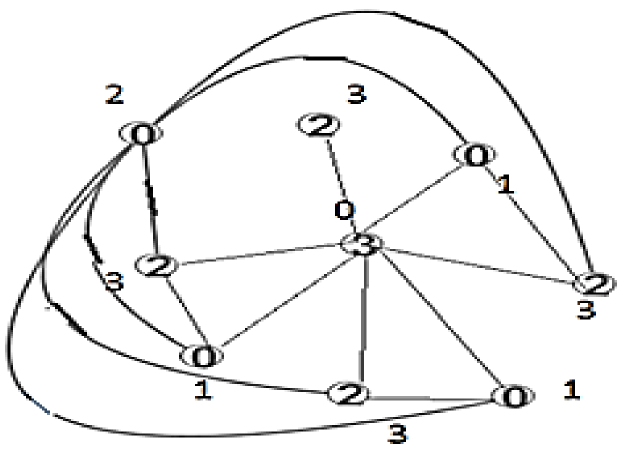

Figure 15. Switching with modular coloring in FSs(4)

4.Modular colorings after switching on Gear graph.

Let $u \in \ell_0$ be the center of a gear graph.Let $v_1,v_3,v_5,\ldots.v_{2n-1}$ be the vertices in $\ell_1$are adjacent to $u \in \ell_0$ and $v_2,v_4,v_6,\ldots.v_{2n}$ be the vertices in ℓ_1.The switching in G(n)is denoted by G$_s$(n).Switching in gear graph is obtained in two ways.
That is (i)switching of the vertex $u \in \ell_0$ and (ii)switching of a vertex $v_i \in \ell_1; i = 1,2,\ldots.,2n$.
(i)By switching of the vertex $u \in \ell_0$ in a gear graph G(n) result in another gear graph G'(n) in which vertices in $\ell_1$which are not adjacent with $u \in \ell_0$ in G(n) become adjacent with G'(n).Therefore G$_s$(n)=G(n)=G'(n).Hence mc[G$_s$(n)]=mc[G(n)]=mc[G'(n)].
(ii) switching of a vertex $v_i \in \ell_1; i = 1,2,\ldots.,2n$.Here specifying the vertex $v_i \in \ell_1$ which are adjacent with $u \in \ell_0$ is taken for switching.In general take v_i as v_1.

Theorem 4.1

The modular coloring of the graph obtained after the switching of a vertex in Gear graph in ℓ_1 (which are adjacent with $u \in \ell_0$).ie (i) mc[$G_s(2)$]=2.(ii) mc[$G_s(n)$]=3;n>2.
Proof.

Case (i) mc[$G_s(2)$]=2.

Switching is taken for $v_1 \in \ell_1$ in G (2).Therefore v_1 is adjacent to v_3 after switching in G_s (n).
Consider a modular coloring c (v):v[$G_s(2)$] → $\mathbb{Z}_2$ defined by $C(v)=\begin{cases}1 & \text{for } v_3 \in \ell_1\\ 0 & \text{elsewhere}\end{cases}$

Then $\sigma(v)=\begin{cases}0 & v_3 \in \ell_1\\ 1 & \text{otherwise}\end{cases}$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in $G_s(2)$
∴mc [G_s(2]=2. Hence the proof.

Case (ii) mc[$G_s(n)$]=3 for n>2.

Switching is taken for $v_1 \in \ell_1$ in G(n).Therefore v_1 is adjacent to v_3 ,v_4,….v_{2n-1} after switching in $G_s(n)$ and not adjacent to the remaining vertices in $G_s(n)$.
Consider a modular coloring c(v):v[$G_s(n)$] → $\mathbb{Z}_3$ defined by

$C(v)=\begin{cases}1 & \text{for } u \in \ell_0;\ v_1 \in \ell_1\\ 0 & \text{elsewhere}\end{cases}$

then $\sigma(v)=\begin{cases}2 & \text{for } v_{3+2j} \in \ell_1\,; j = 0,1,2,\ldots.(n-2).\\ 1 & \text{for } v_{4+2j} \in \ell_1\,; j = 0,1,2,\ldots.(n-3)\\ 0 & \text{otherwise}\end{cases}$ Here $\sigma(x) \neq \sigma(y)$ ∀ x,y of adjacent vertices in $G_s(n)$ ∴ mc[$G_s(n)$]=3 for n>2. Hence the proof.
Eg:

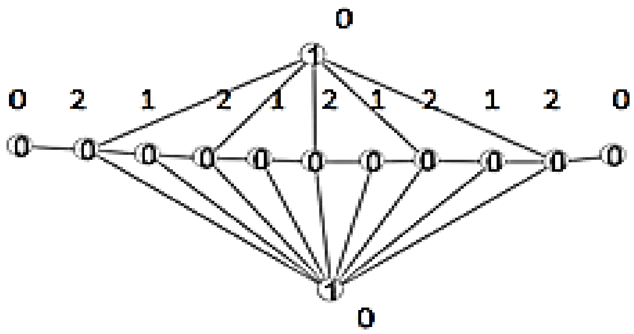

Figure 16. Switching with modular coloring in Gs(6).

5. Conclusions

In a Wheel Graph the modular coloring of a graph obtained after the switching of a vertex $v_i \in \ell_1$ is $W_s(n)=3$ for n=4k,4k+1,[4k+2;k>1]; $W_s(6)=4$; $W_s(n)=4$ for n=4k+3,k≥1.The labeling is quite similar to one other and differs according to the change in number of vertices. Also in a Friendship Graph the modular coloring after switching a vertex in ℓ_1 then (i)mc[$FS_s(2)$]=3. (ii) mc[$FS_s(n)$]=4;n≥ 3. Similarly in Gear Graph the modular coloring obtained after the switching of a vertex in ℓ_1 (which are adjacent with u $\in \ell_0$).ie (i) mc[$G_s(2)$]=2.(ii) mc[$G_s(n)$]=3;n>2. Altogether It is explicitly clear that after switching in different levels of the graphs, the modular chromatic number varies in between two to four. We cannot expect a higher level of modular chromatic number after switching in vertices at different levels. Studying this problem and related problems in the context of switching graphs may help in answering the long open question whether all of these problems have a polynomial algorithm. We conclude this paper by listing a number of switching graph problems of which we do not know the complexity

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