Onset of Turbulence in Planar and Circular Pipe

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ABSTRACT

A third-order hydrodynamic equation with a molecular structure parameter, obtained through a projection and perturbation formalism from the Liouville equation is applied to circular and planar Poisseuille-Hagen flow. It is shown that there is no principal difference in the resulting parabolic velocity profiles as long as the flows remain laminar. However, a difference is noted in the onset of turbulence in consistency with observations, showing larger stability of the parabolic velocity profile in circular pipe.

INTRODUCTION

The origin of turbulence and laminar-turbulent transition are among the most important unresolved problems of fluid dynamics (Fasel & Saric, 1999). The traditional method in standard hydrodynamics is to solve Navier-Stokes equation for stationary velocity profiles, which are parabolic for Poisseuille-Hagen flow in laminar regime in good agreement with experiments. However, any agreement breaks down at the onset of turbulence as velocity profiles flatten and become non-stationary (Landau & Lifshic, 1988; Fox & Germano, 1961). An explanation for the flattened velocity profiles was given by Prandtl and von Karman (Fox & Germano, 1961) using the transverse component of the fluctuation of the velocity. The faster molecules of the central region of the pipe show up in the boundary layer mix with slower molecules, and the velocity profile becomes roughly uniform except boundary layers.

For the description of turbulent flows, many renormalized perturbation theories all based on Reynolds equation (McComb, 1992) were developed in the past. The Reynolds equation, which aside from the mean velocity considers its arbitrary time fluctuations, like Navier-Stokes equation, ignores the structure of the molecules. Limitations of the Reynolds equation in the laminar-turbulent transition and its difficulty in explaining the origin of turbulence suggest study of the phenomenon directly from the Liouville equation.

From the Liouville equation we derive third order hydrodynamic equation containing two control parameters related to the internal structure and geometry of the molecule (Muriel & Dresden, 1995). For the incompressible flows, we then obtain corresponding flow equations for planar and circular pipe. The flow equations with just one control parameter related to the internal structure of the molecule are solved numerically for the velocity profiles for each system separately to see the differences in the onset of turbulence and the change from the parabolic velocity profile into the flattened one. The control parameter is interpreted as the strength of inelastic interactions among the molecules of the fluid. The inelasticity of the collisions is easily explained if one adopts a hypothesis of quantum origin of turbulence (Muriel & Dresden, 1995). There are dissipative effects due to the excitations of internal degrees of freeedom of the molecules, inducing the occurence of inelastic interactions and irregular motion. The quantum kinetic model of turbulence (Muriel & Dresden, 1995) is based

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on the well-known Landau idea (Landau & Lifshic, 1988), which associates turbulence with deterministic chaos.

Hydrodynamic equations

Following the projection formalism of Zwanzig, Dresden, and Muriel (Zwansig, 1961; Muriel & Dresden, 1969) and the perturbation procedure outlined in Jirkovsky & Muriel (1994), the kinetic equations correct to the appropriate order $k = 0,1,2$ are derived from the Liouville equation for N-particle distribution function:

$$i \; \frac{\partial f^{(N)}}{\partial t} = L f^{(N)} \tag{1}$$

with Liouville operator $L = L_o + \lambda L_i$, where

$$L_o = -i \sum \frac{p_i}{m} \vec{\nabla}_{ri} \quad \text{and} \quad L_i = \frac{1}{2} i \sum_{j \neq j} \vec{\nabla}_{rj} V_{jj} (\vec{\nabla}_{pj} - \nabla_{pj})$$

define a projector $P = \dfrac{1}{\Omega^{N-1}} \int ... \int d\vec{r}_2 ... d\vec{r}_N$, where Ω is the volume of the system. The complementary projector is $1 - P$. Applying both projectors to the Liouville equation, we reduce it to an exact equivalent of the first equation in the BBGKY hierarchy for the one particle distribution function

$$\frac{\partial f}{\partial t} = -iL_o f - i\lambda PL_i G(1-P) f^{(N)}(0)$$
$$- \lambda_2 PL_i \int_o^t G(t-s) L_i f(s) ds \tag{2}$$

We then take a very simple approach: formally expand the distribution function $f = \sum_{n=0}^{\infty} \lambda^k f^{(k)}$ in orders of λ

and the propagator $G = e^{i(1-P)Lt}$ in Taylor series, and then substitute into Eq. (2). We pick the terms to the appropriate order k.

Zero order (k=0)

Zero order kinetic equation for the one-particle distribution function $f^{(0)}$ is the Boltzmann equation, with BBGKY-like elastic collision term

$$\frac{\partial f^{(0)}}{\partial t} + \frac{p}{m} \vec{\nabla} f^{(0)} = \left(\frac{\partial f^{(0)}}{\partial t} \right)_{coll} \tag{3}$$

in the absence of an external field, reducible to the standard hydrodynamic equations.

First order (k=1)

We obtain a hydrodynamic equation with one correction term $\partial j < UiUj > t$, known as the Reynolds equation. Corresponding Fourier transformed form is used in almost all theories of turbulence. The turbulence is considered merely as a flow phenomenon and a mathematical problem, molecular structure being unimportant or unnecessary. Although some theories, such as the Kolmogorov theory (McComb, 1992), correctly predict the power spectrum, and fully developed turbulence is considered as well understood, the question of laminar-turbulent transition and origin of turbulence, on the other hand, is still open.

Second order (k=2)

The second order kinetic equation is reduced to the equation for the mean velocity of the fluid   using a renormalization attributed to McComb (McComb, 1992)

$$\rho \left(\frac{\partial \vec{U}}{\partial t} + (\vec{U}\vec{\nabla})\vec{U} \right) + \vec{\nabla} p \; - \; \rho \upsilon \vec{\nabla}^2 \vec{U} =$$
$$- \frac{1}{m} \int_0^t b(t-s) \vec{\nabla} \rho \, ds \; - \; \frac{1}{m} \int_0^t b(t-s)^2 \vec{\nabla}^2 \vec{U} ds \tag{4}$$

where p is the mean pressure, ρ is the density, and ν is the kinematic viscosity.

The control parameter is $b = \int\limits_{(\Omega)} (\vec{\nabla} V)^2 d\Omega$, with Ω as

the volume and *V*, the interaction potential. We have $b=0$ for elastic collisions. This suggests the interpretation of $b \geq 0$ as a measure of the strength of inelastic interactions. The parameter *b* manifests its presence in the turbulent regime and an application such as the planar Poisseuille-Hagen flow (Jirkovsky & Bo-ot, 1999) is used to illustrate the effect of *b*. Numerical simulation has shown time development of parabolic velocity profiles into nearly uniform velocity profiles indicative of the onset of instability. An adhoc assumption was utilized, specifically, the control parameter was increased linearly with the mean velocity. Also, there was a short time limitation in the validity of the second order equations. For large times, the velocities changed signs.

Third order (k=3)

To obtain the third order kinetic equation, we pick terms containing λ^3 in Eq. (2) using the appropriate expansions for the one-particle distribution function and propagator *G*. The third order kinetic equation can be written as:

$$
\frac{\partial f^{(3)}}{\partial t} = -iL_o f^{(3)} - PL_i \int_0^t \{ \frac{1}{6} i(t-s)^3 [(L_o^2 L_i^2
$$
$$
+ L_o L_i L_o L_i + L_i L_o^2 L_i) f^{(0)}(s) + L_o^3 L_i f^{(1)}(s)]
$$
$$
- [i(t-s)L_o L_i - \frac{1}{2}(t-s)^2 L_o^2 L_i] f^{(1)}(s) \} ds
$$

$$(5)$$

Using explicit forms of the Liouville operators, multiplying Eq. (5) by one component of momentum and integrating over velocity space, it may be reduced to momentum transport equation of the third order:

$$
\rho \left[\frac{\partial \vec{U}}{\partial t} + (\vec{U}\vec{\nabla})\vec{U} \right] + \vec{\nabla}p - \upsilon\rho\vec{\nabla}^2\vec{U} =
$$
$$
- \frac{1}{m} \int_0^t b[(t-s)\vec{\nabla}\rho + (t-s)^2\vec{\nabla}^2\vec{U}] ds
$$
$$
- \frac{4}{m^2} \int_0^t (t-s)^3 [5\vec{c}\vec{\nabla}^2\rho + b\vec{\nabla}^3\vec{U}^2] ds \qquad (6)
$$

where the vector parameter $\vec{c} = \int_{(\Omega)} (\vec{\nabla}V)^3 d\Omega$

is related to the geometry of the molecule and is interpreted as a measure of the assymetry of the molecule. We used the renormalization of the density, pressure, and velocity to their true values attributed to McComb (McComb, 1992). Eq. (6) is the hydrodynamic equation with correction terms in integral form correct to the third order. It contains the second order equation and new terms with vector parameter $\vec{c}$ and scalar parameter *b*. It is interesting to note that the geometry of the molecule plays no role in the flow of an incompressible fluid, although there is another term with control parameter *b*.

Planar and circular Poisseuille-Hagen flow

If one uses the incompressibility condition, Eq. (6) may be reduced to non-linear equation for the Poisseuille-Hagen flow, with configuration where $\vec{U} = (U(z,t),0,0)$ for the fluid enclosed between two infinite parallel planar boundaries at fixed distance L:

$$
\frac{\rho}{\upsilon} \frac{\partial^5 U}{\partial t^5} - \frac{\partial^6 U}{\partial t^4 z^2} - \frac{2b}{m\upsilon} \frac{\partial^3 U}{\partial t \partial z^2}
$$
$$
- \frac{4!2b}{m^2} \left(3\frac{\partial U}{\partial z} \frac{\partial^2 U}{\partial z^2} + U\frac{\partial^3 U}{\partial z^3} \right) = 0 \qquad (7)
$$

Utilizing a similar procedure for the flow in infinite circular pipe of radius *R*, with configuration where $\vec{U} = (0,0,U(r,t))$ in cylindrical polar cordinates we have:

$$
\frac{\rho}{\upsilon} r^2 \frac{\partial^5 U}{\partial t^5} - r^2 \frac{\partial^6 U}{\partial r^2 \partial t^4} - r\frac{\partial^5 U}{\partial r \partial t^4}
$$
$$
- \frac{2b}{m\upsilon} \left(r^2 \frac{\partial^3 U}{\partial r^2 \partial t} + r\frac{\partial^2 U}{\partial r \partial t} \right)
$$
$$
+ \frac{2b4!}{m^2} \left[\begin{array}{c} 3r^2 \frac{\partial U}{\partial r} \frac{\partial^2 U}{\partial r^2} - r\left(\frac{\partial U}{\partial r}\right)^2 \\ + r^2 U \frac{\partial^3 U}{\partial r^3} - U\frac{\partial U}{\partial r} - rU\frac{\partial^2 U}{\partial r^2} \end{array} \right] = 0 \qquad (8)
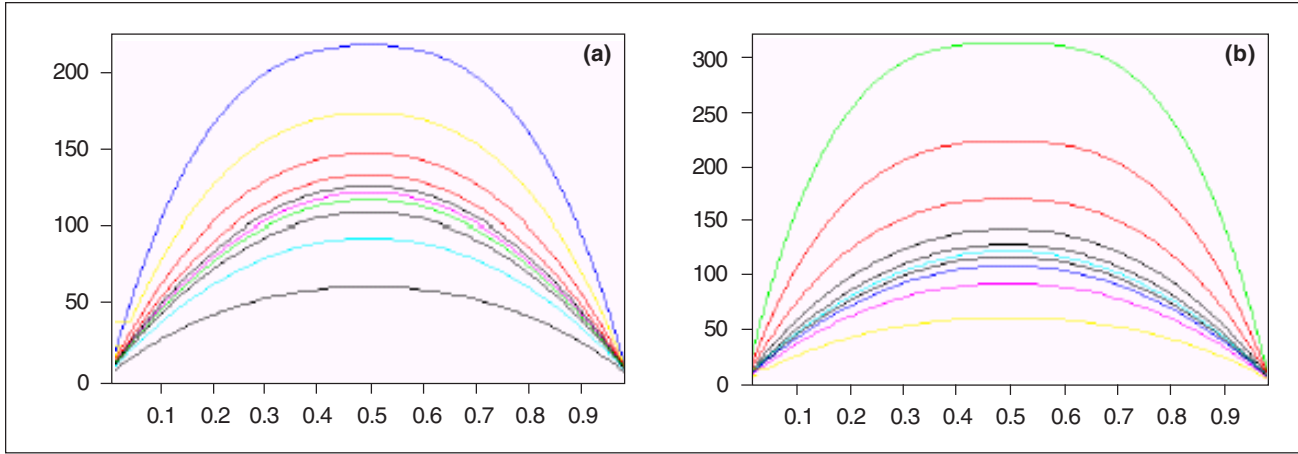$$

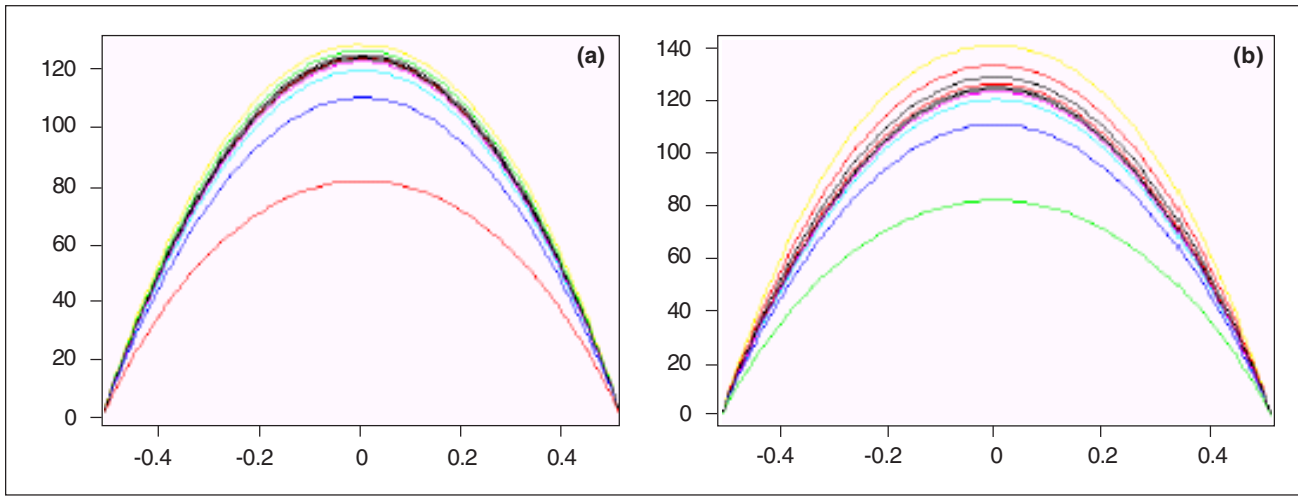Fig.1. Bottom to top, planar flow: (a) b = 0.00000005, t = 1.10; (b) b = 0.0000001, t = 1.10.

Fig. 2. Bottom to top, circular pipe flow: (a) b = 0.00000005, t = 1.10; (b) b = 0.0000001, t = 1.10.

We use non-slip boundary conditions and static fluid as initial conditions to get numeric solutions of standard Navier-Stokes equations, which are subsequently used as initial guess in simulation of third order hydrodynamic equations for the flow in planar and circular pipe (Figs. 3 & 4). Fixed parameters for the numeric simulation are as $L = 1$, $R = 0.5$, $\rho = 1$, $v = 0.1$, and $m = 1$.

Since constant pressure gradients drive the motion of the flows, they are adjusted in standard Navier-Stokes equations to give equal maximum velocity and Reynolds number of the flow in planar and circular pipe. The Reynolds number is $R_n = \dfrac{U_{max}l}{v}$ with characteristic length $l = 0.5L = R$.
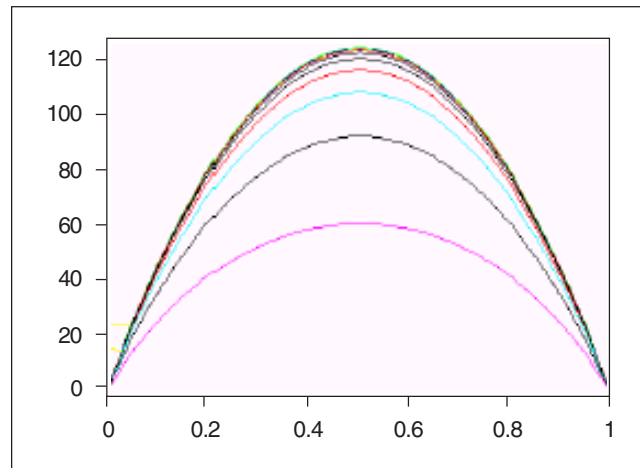

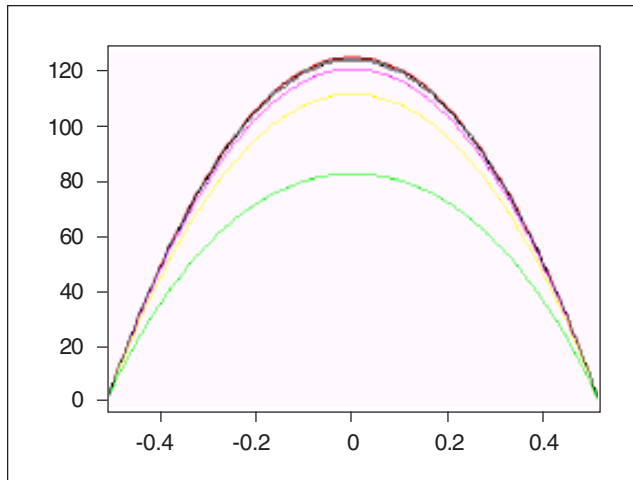
Fig.3. Bottom to top, circular pipe flow: b = 0, t = 1.10.

Fig. 4. Bottom to top, planar flow: b = 0, t = 1.10.

The numeric results (Figs. 1a & 1b) for planar Poisseuille-Hagen flow show the onset of instability and a flattened velocity profile for control parameter b = 0.00000005. On the other hand, velocity profiles in circular pipe with the same control parameter (Figs. 2a & 2b) are still parabolic and the flow is laminar, though quasi-stationary. Higher control parameter is needed to induce laminar-turbulent transition. This is consistent with the observation (Landau & Lifshic, 1988) of critical Reynolds numbers for the circular pipe, Rc = 1800, and for the planar flow, Rc = 1000. Results for the third order theory represent improvement over second order in the time validity and shape of the profiles. Also, velocities do not change signs for large times.

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