Science and Technology, 2 (1997) 57-64

Fourth-Order Approximation of the Fundamental Matrix of a Linear System of Differential Equations

A. M. Hussein

#Department of Computer Science, College of Science, Sultan Qaboos University, P.O.Box 36, Al-Khod 123, Muscat, Sultanate of Oman.

تقريب من الدرجة الرابعة للمصفوفة الأساسية لمنظومة معادلات تفاضلية خطية

عبدالله محمد حسين

خلاصة: هذا البحث يقدم صيغة تقريبية جديدة من الرتبة الرابعة على شكل مصفوفة أسية تمثل مصفوفة الحل الأساسية لمعادلات تفاضلية خطية. تحسب قيمة المصفوفة الأسية من خلال تجزاة مدى التفاضل إلى مناطق جزئية واستخدام تقريب «بادي» مع مبدأ «التوليف والتربيع». يمكن استخدام الصيغة الجديدة المقترحة لحل مسائل القيمة الإبتدائية أو مسائل القيمة المحدودة ذات القيم الحدودية المختلطة. كذلك يمكن استخدام المصفوفة التقريبية المقترحة كأداة تكامل لطرق الحل التي تتطلب معلومات عن القيم على مدى التكامل. يتعرض البحث لمثال عن استخدام التقريب المقترح لحل احدى مسائل القيمة المحدودة ذات القيم الحدودية المختلطة.

ABSTRACT: A fourth-order approximation to the fundamental matrix of a system of linear differential equations is presented in closed form as a matrix exponential. The matrix exponential is then discretized over the interval of integration. Pade approximation together with the method of scaling and squaring (Moler *et al*, 1978) is used to evaluate the matrix exponential. This approach is suitable for solving both initial and boundary value problems with mixed boundary conditions. The approximating matrix can also be used as an integrating operator for methods which require information about the solution along the discretized subintervals. An example of a boundary value problem with mixed boundary conditions is presented.

The problem being considered is governed by the following set of linear differential equations

$$\mathbf{y}'(x) = A(x)\mathbf{y}(x), \quad x \in [a, b] \tag{1}$$

where $\mathbf{y}$ is a vector-valued function with n components, A is an n x n coefficients matrix. The motivation to study problem (1) comes from the fact that it includes various practical problems, including those arising in optimal control theory. The problem of obtaining a closed-form analytical representation of the fundamental matrix, has been extensively studied with limited success. However, if A is commutative then the fundamental matrix can be written in closed-form as a matrix exponential function, (Zhu *et al* 1992). Shridharan *et al* (1995) developed the existence-uniqueness as well as constructive theory for the solution of systems of nonlinear boundary value problems when only approximations of the fundamental matrix of the associated homogeneous linear differential systems are known. Roberts (1979) reported a method that directly uses the fundamental matrix to solve two-point boundary value problems with implicit boundary conditions. That method is based on the Alspaugh-Kagiwada-Kalaba method (Alspaugh *et al* 1970) of invariant imbedding, where the fundamental matrix and its partitions are related to the Riccati equation.

The solution of equation (1) is given by

$$\mathbf{y}(x) = \Phi(x, a)\mathbf{y}(a)$$

where $\Phi(x, a)$ is the fundamental matrix, satisfying $\Phi(a, a) = I$, and I is the n x n identity matrix. One way of evaluating $\Phi(x, a)$ is by finding successive approximations to the solution in the form of the infinite series, (Coddington et al, 1955)

$$\Phi(x,a) = I + \int_a^x A(s_0)\,ds_0 + \int_a^x A(x_0)\int_a^{s_0} A(s_1)\,ds_1\,ds_0$$

$$+ \int_a^x A(s_0)\int_a^{s_0} A(s_1)\int_a^{s_1} A(s_2)\,ds_2\,ds_1\,ds_0 + \ldots$$

However, this requires the evaluation of multiple integrals of increasing complexity if accurate answers are to be found. Asfar *et al* (1989a) suggested the division of the interval $[a, b]$ into a number of subintervals $[x_j, x_{j+1}]$, $j = 0, 1, \ldots, (L-1)$, where x_0 = a and x_L = b. The matrix $A(x)$ is then approximated by a constant matrix A_j over each subinterval $[x_j, x_{j+1}]$, and $\exp((x - x_j) A_j)$ was used to approximate $\Phi(x, x_j)$ in that subinterval. In that approach $\exp((x - x_j) A_j)$ is a first-order approximation to $\Phi(x, a)$, and was used successfully by Asfar *et al* (1989b) and by Hussein (1994). In this work, higher order approximations to the matrix exponential are presented.

The fundamental matrix $\Phi(x, a)$ satisfies the equation

$$\Phi'(x, a) = A(x)\Phi(x, a), \tag{2}$$

so that expanding $\Phi(x, a)$ about $x = a$ in powers of $x - a$ and substituting in equation (2) gives

$$\Phi(x, a) = \sum_{j=0}^{\infty} \frac{(x-a)^j}{j!}\Phi^{(j)}(a, a) \tag{3}$$

where $\Phi^{(j)}(a, a)$ is the jth derivative of $\Phi(x, a)$ with respect to x, evaluated at $x = a$. Differentiating equation (2) repeatedly give the matrices $\Phi^{(j)}(a, a)$ as follows

$\Phi(a, a) = I$

$\Phi'(a, a) = A_0$

$\Phi''(a, a) = A_0^2 + A_0'$

$\Phi'''(a, a) = A_0^3 + A_0A_0' + 2A_0'A_0 + A_0''$

and so on, where $A_0 = A(a)$, $A_0' = A_0'(a)$, $A_0'' = A''(a)$, . . . etc.

Substituting the above values of Φ and their derivatives in equation (3) and grouping equal powers of $x - a$, gives

$$\Phi(x,a) = \exp((x-a)A_0) + \frac{(x-a)^2}{2!}A_0' + \frac{(x-a)^3}{3!}[A_0A_0' + 2A_0'A_0 + A_0''] + \frac{(x-a)^4}{4!}[A_0^2A_0' + 2A_0A_0'A_0 + A_0A_0'' + 3A_0'A_0^2 + 3A_0'A_0' + 3A_0''A_0 + A_0'''] + \ldots \tag{4}$$

In equation (4), the expression $\exp((x-a)A_0)$ is a first-order approximation to $\Phi(x, a)$. Higher order approximations to the solution can be obtained, using the approximation

$$\Psi(x, a) = \exp\left(t\sum_{j=1}^{r} \alpha_j A(\alpha + \beta_j t)\right), \tag{5}$$

for $\Phi(x, a)$ where $t = x - a$ and $r = 2$. Expanding $A(a + \beta_j t)$ in powers of t, substituting in equation (5) and grouping equal powers of t, give

$$\Psi(x,a) = \exp\left(\sum_{j=0}^{\infty} u_j \frac{t^j}{j!} A_0^{(j)}\right), \quad u_j = \sum_{i=1}^{r} \alpha_i \beta_i^j \tag{6}$$

Expanding the exponential function in equation (6), and grouping equal powers of t, give

$$\Psi(x,a) = \exp((x-a)u_0A_0) + \frac{2u_1}{2!}(x-a)^2A_0' + \frac{(x-a)^3}{3!}[3u_0u_1(A_0A_0' + A_0'A_0) + 3u_2A_0''] + \ldots$$

If $u_0 = 1$ and $u_1 = ½$, then the difference $\mathcal{E}(x, a)$ between $\Phi(x, a)$ and $\Psi(x, a)$ is

$$\mathcal{E}(x,a) = \Phi(x,a) - \Psi(x,a) = \frac{(x-a)^3}{12}[A_0'A_0 - A_0A_0' + 6(\frac{1}{3} - u_2)A_0''] + O((x-a)^4).$$

With $u_0 = 1$ and $u_1 = ½$, the parameters α_1, α_2, β_1 and β_2 satisfy the following equations

$$\alpha_1 + \alpha_2 = 1 \tag{7}$$

$$\alpha_1\beta_1 + \alpha_2\beta_2 = ½ . \tag{8}$$

The equations (7) and (8) can be used to generate a second-order family of approximations to $\Phi(x, a)$ with two free parameters, say α_1 and β_1. In order to keep the number of matrix $A(x)$ evaluations to a minimum, (by evaluating $A(x)$ only on the discretized points) the values of the free parameters are taken to be $\alpha_1 = ½$, $\beta_1 = 0$, so that $\alpha_2 = ½$ and $\beta_2 = 1$. Thus

$$\Psi(x, a) = \exp((x-a)\{A(a) + A(x)\}/2) \tag{9}$$

and

$$\mathcal{E}(x,a) = \frac{(x-a)^3}{12}[A(a)A(a) - A(a)A'(a) - A''(a)] + O((x-a)^4). \tag{10}$$

Also one can take $\alpha_1 = 0$, so that $\alpha_2 = 1$ and $\beta_2 = ½$ and

$$\Psi(x, a) = \exp\left((x-a)A(\frac{a+x}{2})\right) \tag{11}$$

with

$$\mathcal{E}(x,a) = \frac{(x-a)^3}{12}[A'(a)A(A) + \frac{1}{2}A''(a)] + O((x-a)^4)). \tag{12}$$

The $\Psi(x, a)$ matrix given in equations (9) and (11) is a second-order approximation to the fundamental matrix $\Phi(x, a)$, with the error $\mathcal{E}(x, a)$ given by equations (10) and (12), respectively.

Fourth-order approximation

Taking $r = 3$ in equation (5) gives,

$$\begin{aligned}\mathcal{E}(x,a) = \frac{(x-a)^3}{12}&[A'(a)A(a) - A(a)A'(a)\\ &+ 2(1-3u_2)A''(a)] + O((x-a)^4)\end{aligned}$$

where $u_0 = 1$ and $u_1 = ½$. Thus increasing the value of r in equation (5) does not increase the order of approximation. However we could use the approximation $\Omega(x, a) = \exp(t B_1 + t^2 B_2)$, where $t = x - a$,

$$B_1 = \sum_{j=1}^{3} \alpha_j A(a+\beta_j t), \quad B_2 = \sum_{j=1}^{3} \gamma_j A(a+\delta_j t).$$

Expanding A in powers of t and grouping equal powers, gives

$$\begin{aligned}\mathcal{F}(x,a) &= \Phi(x,a) - \Omega(x,a) = e^{(x-a)A_0} - e^{u_0(x-a)A_0}\\
&+ \frac{(x-a)^2}{2!}[(1-2u_1)A_0' - 2(\gamma_1+\gamma_2)A\\
&+ \frac{(x-a)^3}{3!}[(1-3u_2)A_0'' + \{2 - 3u_0u_1\\
&- 6(\gamma_1\delta_1+\gamma_2\delta_3)\}A_0'A_0\\
&+\{1 - 3u_0u_1 - 6(\gamma_1\delta_2+\gamma_2\delta_4)\}A_0A_0'\\
&- 6u_0(\gamma_1+\gamma_2)A_0^3]\\
&+ \frac{(x-a)^4}{4!}[\{1 - 4u_0^2u_1 - 12u_1(\gamma_1+\gamma_2\\
&- 12u_0(\gamma_1\delta_2+\gamma_2\delta_4)\}A_0^2A_0'\\
&+\{3 - 4u_0^2u_1 - 12(\gamma_1+\gamma_2) - 12u_0(\gamma_1\\
&+\gamma_2\delta_3)\}A_0'A_0^2\\
&+(1-4u_3)A_0''' - 12\{(\gamma_1+\gamma_2)^2\\
&+u_0^2(\gamma_1+\gamma_2)\}A_0^4\\
&+\{2-12u_0(\gamma_1\delta_1+\gamma_2\delta_3)-12u_0(\gamma_1\delta_2+\gamma_2\\
&-4u_0^2u_1\}A_0A_0'A_0\\
&+\{1 - 6u_0u_2 - 12(\gamma_1\delta_2^2+\gamma_2\delta_4^2)\}A_0A_0''\\
&+\{3 - 6u_0u_2 - 12(\gamma_1\delta_1^2+\gamma_2\delta_3^2)\}A_0''A_0\\
&+\{3 - 12u_1^2 - 24(\gamma_1\delta_1\delta_2+\gamma_2\delta_3\delta_4)\}A\\
&+O((x-a)^5).\end{aligned}$$

If

$$u_0 = 1,\ u_1 = \frac{1}{2},\ u_2 = \frac{1}{3},\ u_3 = \frac{1}{4}. \tag{13}$$

$$\gamma_1+\gamma_2=0,\ \gamma_1\delta_1+\gamma_2\delta_3=\frac{1}{12},\ \gamma_1\delta_2+\gamma_2\delta_4=\frac{-1}{12}. \tag{14}$$

$$\begin{aligned}&\gamma_1\delta_1^2+\gamma_2\delta_3^2=\frac{1}{12},\ \gamma_1\delta_2^2+\gamma_2\delta_4^2=\frac{-1}{12},\\ &\gamma_1\delta_1\delta_2+\gamma_2\delta_3\delta_4=0.\end{aligned} \tag{15}$$

then $\mathcal{F}(x, a) = O((x - a)^5)$. The parameter values given in equations (14) and (15) result in the following relations

$$\delta_1 = \frac{12\gamma_1+1}{24\gamma_1} = \delta_4, \quad \delta_2 = \frac{12\gamma_1-1}{24\gamma_1} = \delta_3.$$

The conditions given in equation (13) give the following system of four non-linear algebraic equations, in terms of the parameters α_1, α_2, α_3, β_1, β_2, and β_3.

$$\sum_{j=1}^{3}\alpha_j\beta_j^i = \frac{1}{i+1}, \quad i = 0,\ 1,\ 2,\ 3.$$

In order to keep the number of the coefficient matrix A evaluations to a minimum, the following values are taken for β_1, β_2, β_3 and γ_1:

$$\beta_1 = 0,\quad \beta_2 = \frac{1}{2},\quad \beta_3 = 1,\quad \gamma_2 = \frac{-1}{12},$$

and the other parameters are

$$\delta_1 = 0,\quad \delta_2 = 1,\quad \delta_3 = 1,\quad \delta_4 = 0,$$

$$\alpha_1 = \frac{1}{6},\quad \alpha_2 = \frac{4}{6},\quad \alpha_3 = \frac{1}{6},\quad \gamma_2 = \frac{1}{12}.$$

Hence

$$\Omega(x, a) = \exp((x - a) B_1 + (x - a)^2 B_2) \tag{16}$$

where

$$B_1 = \frac{1}{6}[A(a) + 4A(\frac{a+x}{2}) + A(x)],$$

$$B_2 = \frac{1}{12}(A(x)A(a) - A(a)A(x)],$$

and

$$\mathcal{J}(x,a)=\frac{(x-a)^5}{5!}\Big[\frac{-1}{24}A_0^{(4)}+\frac{1}{6}\{A_0^3A_0'-A_0'A_0^3 +A_0A_0'''-A_0'''A_0+A_0^2A_0''-2_0A_0''A_0+A_0''A_0^2\} +\frac{1}{2}\{A_0A_0'A_0^2-A_0^2A_0'A_0+A_0A_0'A_0'-2A_0'A_0A_0' +A_0'A_0'A_0\}\Big]+O((x-a)^6)$$

which shows that $\Omega(x, a)$ is a fourth-order approximation to $\Phi(x, a)$. In approximating the fundamental matrix $\Phi(x, a)$ over the interval $[a,b]$, we discretize the second-order approximation $\Psi(x, a)$ or the fourth-order approximation $\Omega(x, a)$ over L subintervals $[x_0, x_1]$, $[x_1, x_2]$, . . . , $[x_{L-1}, x_L]$ where $x_0 = a$, $x_L = b$ and $x_{j+1} = x_j + h_{j+1}$, $j = 0, 1, 2, \ldots, (L-1)$. The approximating matrix is then given, over each subinterval, in closed form.

Computation of the matrix exponential

The matrix exponential $\exp(hQ)$ can be approximated by many methods, but the method of scaling and squaring (Moler et al, 1978) is appropriate especially when $h\|Q\|$ is very large, where the norm is taken to be the Euclidean norm. The method is based on the property of the matrix exponential

$$\exp(hQ) = [\exp(hQ/m)]^m \tag{17}$$

where $m = 2^j$. When $h\|Q\|$ is very large, the roundoff error difficulties can be controlled by using a suitable value of j for which $\exp(hQ/m)$ can be reliably and efficiently computed, and then to form $\exp(hQ)$ by repeated squaring of $\exp(hQ/m)$. The matrix $\exp(hQ/m)$ can be satisfactorily computed by the following Padé approximation:

$$\exp(hQ) \approx R_{qq}(hQ) = [D_{qq}(hQ)]^{-1}N_{qq}(hQ), \tag{18}$$

where

$$N_{qq}(hQ)=\sum_{j=0}^{q}P_j(hQ), \quad D_{qq}(hQ)=\sum_{j=0}^{q}(-1)^jP_j(hQ)$$

and

$$P_j(hQ) = \frac{(2q-j)!q!}{(2q)!j!(q-j)!}(hQ)^j.$$

A rather extensive treatment of Padé approximation can be found in Ralston *et al* (1978). If $\exp(hQ/m)$ is to be approximated by $R_{qq}(hQ/m)$, where $m = 2^j$, then the values of j and q have to be chosen. Moler *et al* (1978) reported that if

$$h\|Q\| \le 2^{j-1}$$

then

$$[R_{qq}(hQ/m)]^m = \exp(hQ + Z)$$

where

$$\frac{\|Z\|}{h\|Q\|} \le 8\left(\frac{h\|Q\|}{2^j}\right)^{2q}\left(\frac{(q!)^2}{(2q)!(2q+1)!}\right) \tag{19}$$

The value of j can be calculated from equation (17), and if

$$\frac{\|Z\|}{h\|Q\|} \le \varepsilon$$

then equation (19) can be used to calculate the value of q, as reported in table (1) below, (Moler *et al* 1978) which gives the optimum values of (q, j) associated with $[R_{qq}(hQ/m)]^m$ for a given ε and $h\|Q\|$.

Applications to systems of differential equations

The second-order approximation $\Psi(x, a)$, given in equations (9) or (11), and the fourth-order approximation $\Omega(x, a)$, given in equation (16), can be used to approximate the solution of a linear system of differential equations, with initial or mixed boundary conditions. In the case of an initial value problem, one of the following equations

$$y(x) = \Psi(x, a)\, y(a)$$

or

$$y(x) = \Omega(x, a)\, y(a)$$

can be used to approximate step by step the solution along the interval of interest. In the case of a boundary value problem, the interval $[a, b]$ of interest is divided into L subintervals $[x_j, x_{j+1}]$, $j = 0, 1, 2, \ldots, (L-1)$; and then the solutions

$$y_{j+1}(x) = \exp((x-x_j)\, G_{j+1}(x, x_j))\, y_{j+1}(x_j) \tag{20}$$

are matched at the mid points. In equation (20) above, $G_{j+1}(x, x_j)$ is taken to be

$$G_{j+1}(x, x_j) = \frac{1}{2}\{A(x_j) + A(x)\}$$

for the second-order approximation, and

$$G_{j+1}(x, x_j) = \frac{1}{6}\left[A(x_j) + 4A\left(\frac{x_j + x}{2}\right) + A(x)\right] + \frac{(x - x_j)}{12}[A(x)A(x_j) - A(x_j)A(x)]$$

for the fourth-order approximation. The matched equations together with the boundary conditions, give a system of linear algebraic equations, with band structure which can easily be solved by Gaussian elimination.

Example:

In this example we consider the following two-point boundary value problem with oscillatory coefficients over the interval $[0, T]$:

$$y'(t) = A(t)\, y(t), \tag{21}$$

where

$$A(t) = \begin{pmatrix} \sin(kt) & \cos(kt) \\ \cos(kt) & -\sin(kt) \end{pmatrix}, \quad y(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix},$$

and $T = s\pi/k$, the interval width T can be changed, by changeing the value of s. The boundary conditions are

$$u(0) - v(0) = 1 \tag{22}$$

$$v(T) = 0. \tag{23}$$

If we divide the interval $[0, T]$ into L subintervals, then the solution along each subinterval $[t_j, t_{j+1}]$ is given by the following set of equations

$$y_{j+1}(t) = \exp((t - t_j)\, G_{j+1}(t, t_j))\, y_{j+1}(t_j), \quad j = 0, 1, 2, \ldots, (L-1) \tag{24}$$

The solutions from two adjacent subintervals are equal at the middle point, then equations (24) can be written as

$$y_{j+1}(t_{j+1}) = \exp(hG_{j+1}(t_{j+1}, t_j))\, y_j(t_j), \quad j = 1, 2, 3, \ldots, (L-2) \tag{25}$$

with the solution at the end points

$$y_1(t_1) = e^{hG_1(t_1, t_0)}\, y_1(t_0) = e^{hG_1(t_1, t_0)} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \tag{26}$$

and

$$\begin{pmatrix} u(T) \\ v(T) \end{pmatrix} = y_L(t_L) = e^{hG_L(t_L, t_{L-1})}\, y_{L-1}(t_{L-1}). \tag{27}$$

Using the approximation in equation (18), the above three equations (25) - (27) become

$$N_1 \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = D_1 y_1(t_1)$$

$$N_{j+1} y_j(t_j) = D_{j+1} y_{j+1}(t_{j+1}), j = 1, 2, 3, \ldots, (L-2)$$

$$N_L y_{L-1}(t_{L-1}) = D_L \begin{pmatrix} v(T) \\ v(T) \end{pmatrix}.$$

Adapting the notation $u(t_j) = u_j$ and $v(t_j) = v_j$, for $j = 0, 1, 2, \ldots, L$ together with the boundary conditions given in equations (26) and (27), a system of $2L \times 2L$ linear algebraic equations with the unknowns $\{u_0, u_1, v_1, u_2, v_2, \ldots, u_{L-1}, v_{L-1}, u_L\}$ is formed. In this example, (2,2)-Padé approximation ($q = 2$) was used together with $j = 0$, to approximate the matrix exponential. Table (2) below shows, the number of subintervals needed to solve problem (21) using either the second-order approximation, or the fourth-order approximation for both $s = k = 10$ and $s = k = 40$. Figures (1) - (4) represent, respectively the $u(t)$ and $v(t)$ components of the solution $y(t)$ for the two sets of s and k.

TABLE 1

The optimum values of (q, j).

ε / $h\|Q\|$	10^{-3}	10^{-6}	10^{-9}	10^{-12}	10^{-15}
10^{-2}	(1,0)	(1,0)	(2,0)	(3,0)	(3,0)
10^{-1}	(1,0)	(2,0)	(2,0)	(4,0)	(4,0)
1	(2,1)	(3,1)	(3,0)	(5,1)	(6,1)
10	(2,5)	(3,5)	(4,1)	(5,5)	(6,5)
10^2	(2,8)	(3,8)	(4,5)	(5,8)	(6,8)
10^3	(2,11)	(3,11)	(4,11)	(5,11)	(6,11)
10^4	(2,15)	(3,15)	(4,15)	(5,15)	(6,15)
10^5	(2,18)	(3,18)	(4,18)	(5,18)	(6,18)
10^6	(2,21)	(3,21)	(4,21)	(5,21)	(6,21)

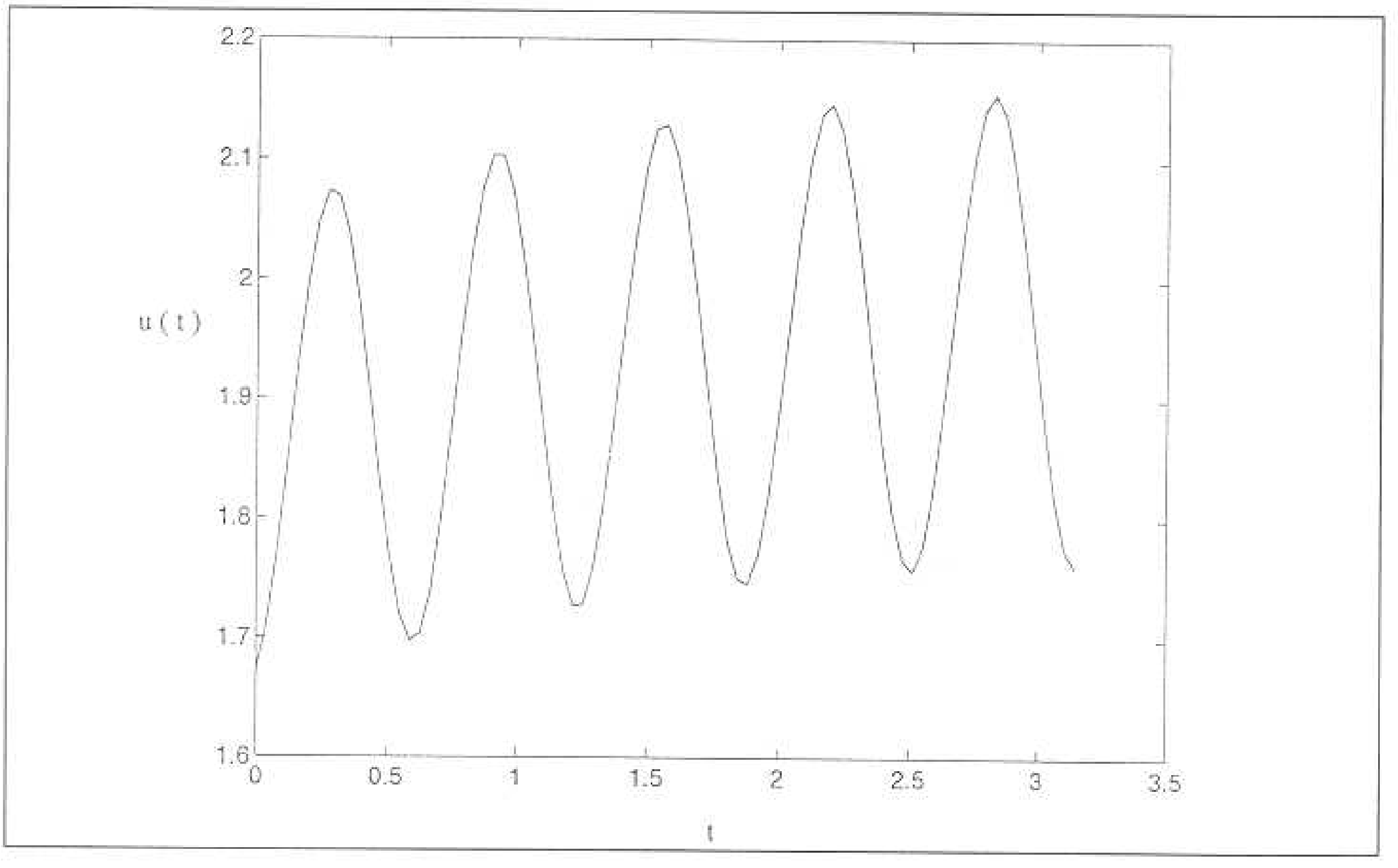

Figure 1. u(t) for s = 10 & K = 10.

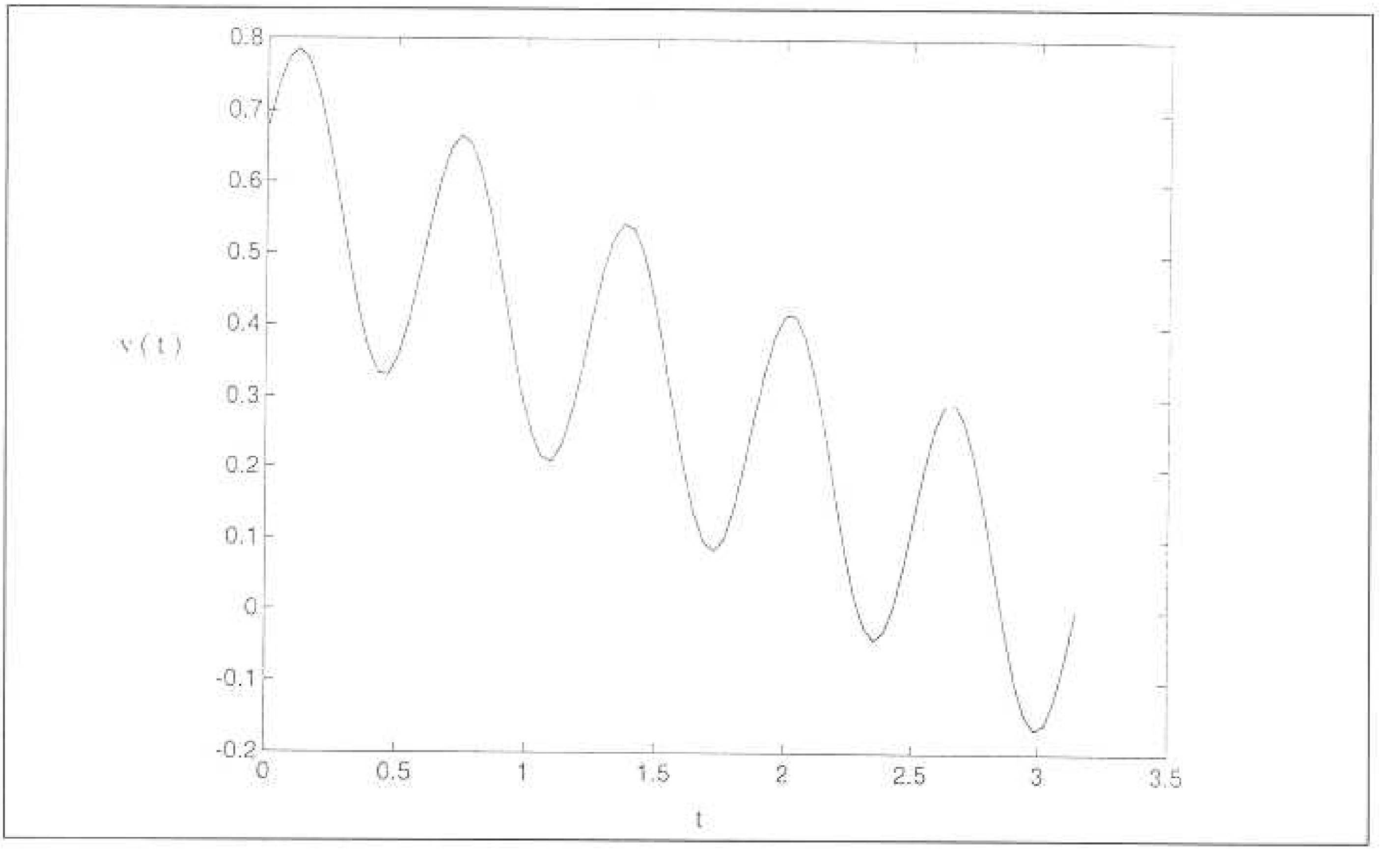

Figure 2. v(t) for s = 10 & K = 10.

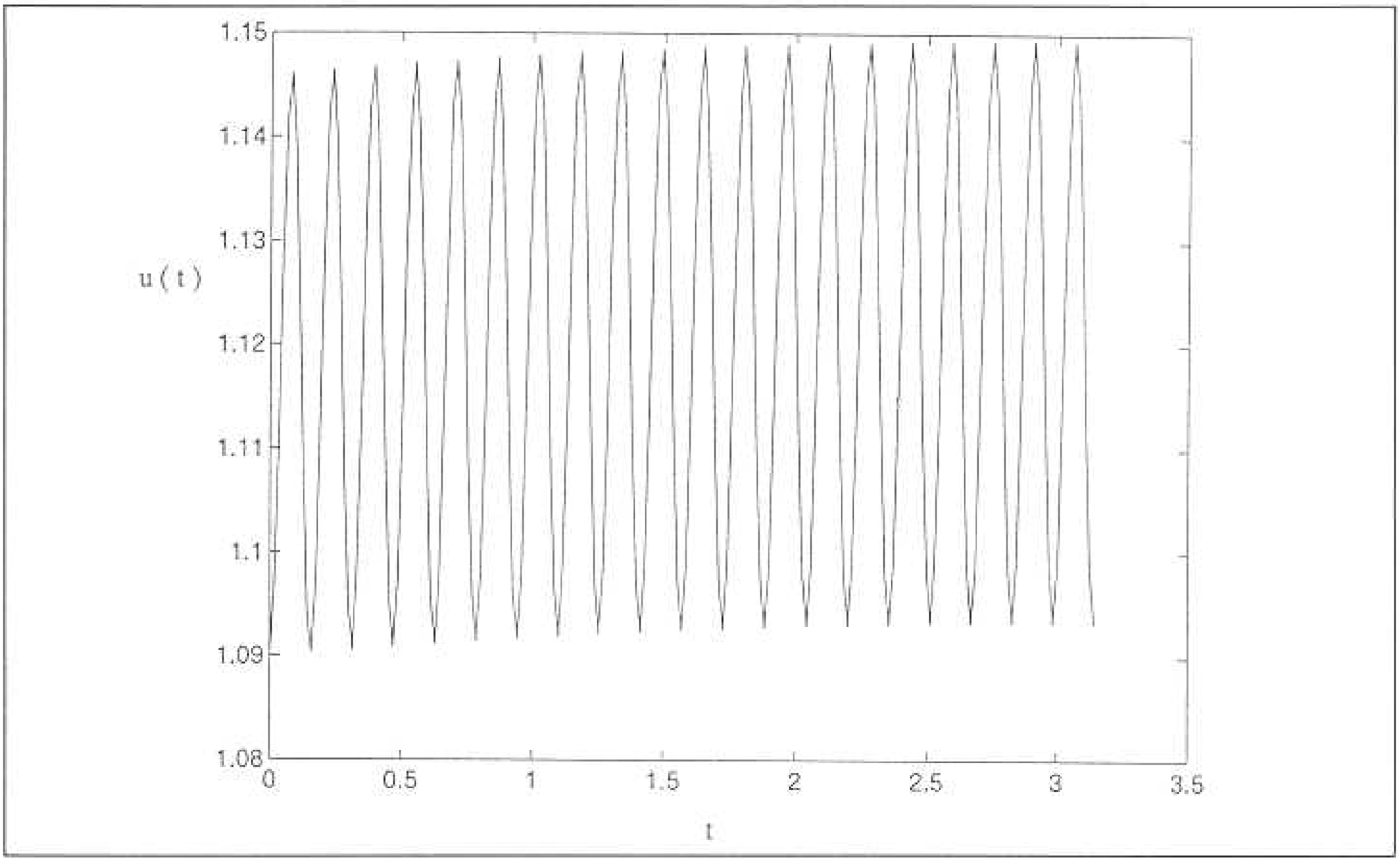

Figure 3. u(t) for s = 40 & K = 40.

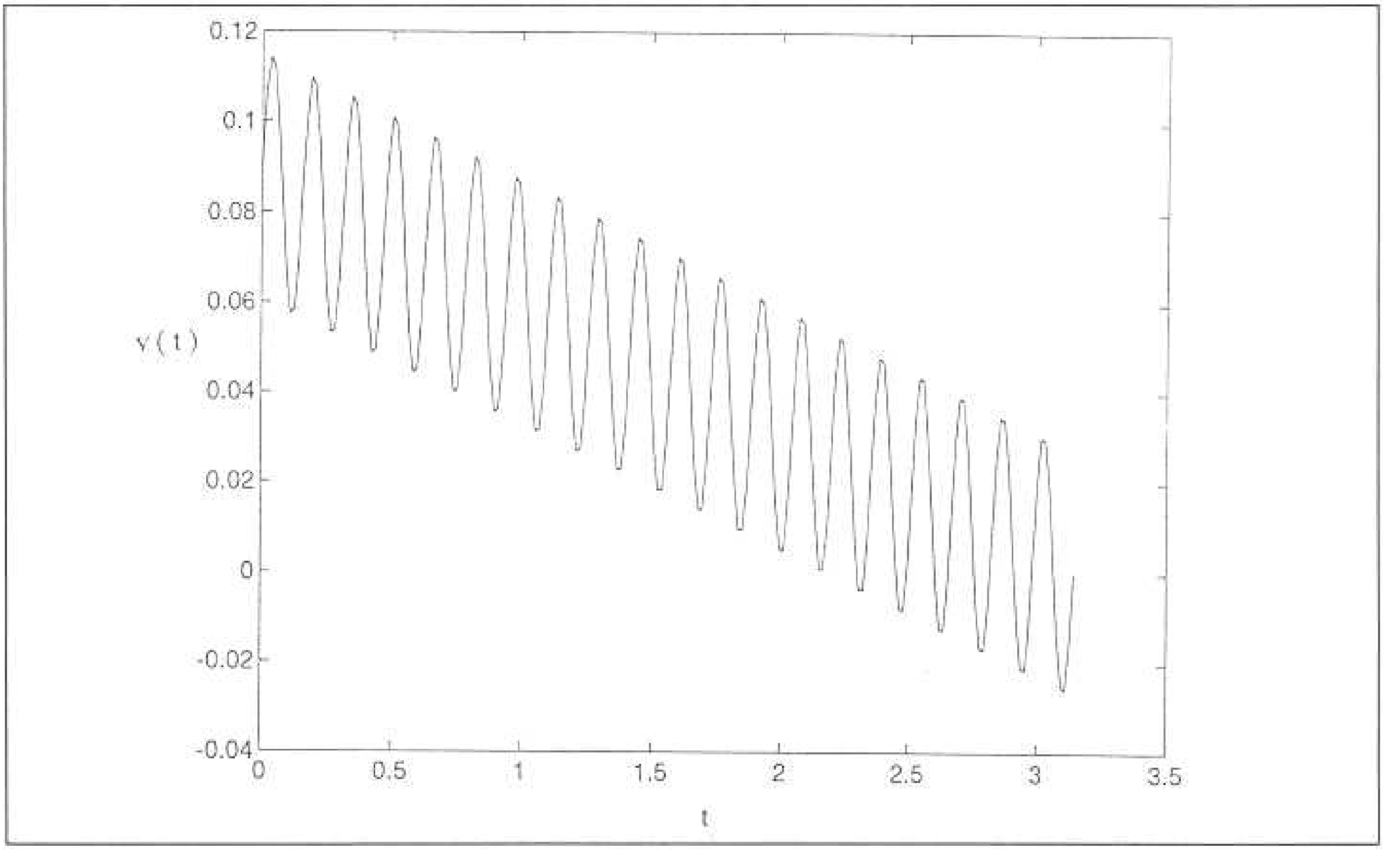

Figure 4. v(t) for s = 40 & K = 40.

TABLE 2
The number of subintervals needed

s	k	2nd-order approximation L for equation (37)	4th-order approximation L for equation (38)
10	10	260	80
40	40	550	200

Conclusion

A fourth-order approximation to the fundamental matrix of a system of linear differential equations was derived in closed form. The derived approximation was successfully applied to a boundary value problem with mixed boundary conditions. The approximation can be easily applied to solve initial value problems as well. The approximation was in the form of a matrix exponential, which was evaluated using Padé approximation together with the method of scaling and squaring.

References

ALSPAUGH D.W., KAGIWADA H.H., and KALABA R., 1970. 'Application of Invariant Invariant Imbedding to the Buckling of Columns', *J. Computational Physics* **5**, pp 56-69.

ASFAR O.R., and HUSSEIN A.M., 1989a. 'Numerical Solution Of Linear Two-Point Boundary Problems Via The Fundamental-Matrix Method', *Int. J. Numer. Meth. Eng.*, **28**, 1205-1216.

ASFAR O.R., HUSSEIN A.M., and IJJAH A., 1989b. 'Application Of The Fundamental Matrix Method For Two-Point Boundary-Value Problems To Mode Coupling In A Parallel-Plate Waveguide Having Multiperiodic Wall Corrugations', *IEEE Trans. Magnet.*, **25**, 2989-299.

CODDINGTON E., and LEVINSON N., 1955. *Theory Of Ordinary Differential Equations*, McGraw - Hill, New York.

HUSSEIN A.M., 1994. 'An Iterative Linearization Of The Non-Linear Inverse Axial Dispersion Model', *Int. Comm. Heat Mass Tran.*, **21**, 217-225.

MOLER C., and VAN LOAN C., 1978. 'Nineteen Dubious Ways To Compute The Exponential Of A Matrix', *SIAM Review*, **20**, 801-836.

ROBERTS S.M., 1979. Direct 'Use of the Fundamental Matrix in Two-Point Boundary Value Problems', *J. Mathematical Analysis Applications* **71**, pp 147-160.

RALSTON A., and RABINOWITZ P., 1978. *A First Course in Numerical Analysis, McGraw-Hill*, New York.

SHRIDHARAN R., and AGARWAL R., 1995. 'General Iterative Methods for Nonlinear Boundary Value Problems', *J. Austral. Math. Soc. Ser.* **B 37**, pp 58-85.

ZHU J., and MORALES C.H., 1992. 'On linear Ordinary differential-Equations with Functionally Commutative Coefficient Matrices, *Linear Algebra and its Applications*, **170**, Iss JUN, pp 81-105.

Received 7 January 1996
Accepted 1 March 1997