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BMOA Characterzation with Families of Cauchy Transforms

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تصنيف BMOA باستعمال مجموعة تحويلات كوشي

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خلاصة : نقدم في هذا البحث عدداً من النتائج المتعلقة بتحويلات كوشي العامة من وإلى القرص الواحدي في مستوى مجموعة الأعداد العقدية باستعمال قياس بورال على المجموعة .

ABSTRACT: In this paper we prove a number of results on Cauchy transforms of generalized type given by Borel measures supported on the class of analytic functions mapping the unit disc into the unit disk.

Let $\Delta = \{z \in C : |z| < 1\}$, $\Gamma = \partial\Delta$ and let B (equipped with the topology of uniform convergence on compact subsets) denote the set of functions ϕ that are analytic in Δ such that $|\phi(z)| < 1$ and $\phi(0) = 0$. Let M, N denote the sets of complex-valued Borel measures on Γ and B respectively. Here, M is equivalent to the subset of N consisting of all those measures supported on the set $\{x \cdot z : |x| = 1\}$.

For $z \in \Delta$ and $\alpha \geq 0$, let A_α denote the family of functions f for which there exists a measure $\mu \in N$ such that

$$f(z) = \begin{cases} \int_B \frac{1}{(1-\phi(z))^\alpha} \mathrm{d}\mu(\phi) & \text{for } \alpha > 0 \\ \int_B \log\frac{1}{1-\phi(z)} \mathrm{d}\mu(\phi) + f(0) & \text{for } \alpha = 0 \end{cases} \tag{1.1}$$

The classes F_α consisting of those functions g for which there exists a measure $\mu \in M$ such that,

$$g(z) = \begin{cases} \int_\Gamma \frac{1}{(1-\bar{x}z)^\alpha} \mathrm{d}\mu(x) & \text{for } \alpha > 0 \\ \int_\Gamma \log\frac{1}{1-\bar{x}z} \mathrm{d}\mu(x) + g(0) & \text{for } \alpha = 0 \end{cases} \tag{1.2}$$

have been well studied (Hallenbeck *et al*, 1996; Hallenbeck and Samotij, 1993; Hruscev and Vinogradov, 1981; Vinogradov, 1980). The classes F_α are subsets of A_α when the measures μ in (1.1) are in M.

The class A_α is a Banach space with respect to the norm

$$\|f\|_{A_\alpha} = \begin{cases} \inf \|\mu\| & , \text{ for } \alpha > 0 \\ \inf \|\mu\| + |f(0)| & , \text{ for } \alpha = 0 \end{cases} \tag{1.3}$$

where μ varies over all measures in N for which the measures μ in (1.1) are in M.

Clearly, for $f \in F_\alpha$, $\|f\| F_\alpha \geq \|f\|_{A_\alpha}$. It is also know form (Brannan *et al*, 1973) that for $\alpha \geq 1$, $F_\alpha = A_\alpha$.

We will show in this paper that for $0 < \alpha < \beta$,

$$A_\alpha \subset A_\beta \text{ and } \|f\|_{A_\beta} \leq \|f\|_{A_\alpha} \tag{1.4}$$

This generalizes similar results for F_α in (Hibschweiler and Nordgren, 1996).

We will also show that $A_0 = BMOA$ and that the norm $\|.\|_{A_0}$ is equivalent to well known BMO norms. Furthermore we will show that, for all $n \geq 0$, $\|z^n\|_{A_\alpha} \leq k$ where the constant k is independent of n or α.

The Classes A_α

In this section we will establish for $0 \leq \alpha < \beta$ the relationship between A_α and A_β as well as their respective norms.

THEOREM 1: If $0 \leq \alpha < \beta$, then $A_\alpha \subset A_\beta$ and $\|f\|_{A_\beta} \leq \|f\|_{A_\alpha}$.

Proof. Note that since $A_\alpha = F_\alpha$ for $\alpha \geq 1$ (Brannan *et al*, 1973), and for $0 < \alpha < \beta$, $F_\alpha \subset F_\beta$ and $\|f\|_{F_\beta} \leq \|f\|_{F_\alpha}$ (Hibschweiler and Nordgren, 1996), then all we have to prove is the case $0 \leq \alpha < \beta < 1$.

(i) Let $f \in A_\alpha$ where $0 < \alpha < \beta$, then we can write

$$f(z) = \int_B \frac{1}{(1-\psi(z))^\alpha} d\mu(\psi). \tag{2.1}$$

Since $\dfrac{1}{(1-z)^\alpha} \in F_\alpha \subset F_\beta$, we can write

$$\frac{1}{(1-z)^\alpha} = \int_\Gamma \frac{1}{(1-\bar{x}z)^\beta} d\nu(x) \tag{2.2}$$

and

$$\left\|\frac{1}{(1-z)^\alpha}\right\|_{F_\beta} \leq \left\|\frac{1}{(1-z)^\alpha}\right\|_{F_\alpha} = 1 \tag{2.3}$$

Now by replacing z in (2.2) by $\psi(z)$ and putting the result in (2.1) we get

$$f(z) = \int_B \int_\Gamma \frac{1}{(1-\bar{x}\psi(z))^\beta} d\nu(x) d\mu(\psi). \tag{2.4}$$

Suppose, without loss of generality that ν is a positive measure and let

$$g_n(z) = \sum_{k=1}^{n} \frac{\nu_k}{(1-\bar{x}_k z)^\beta}.$$

Then by (2.2)

$$\int_B g_n(\psi)d\mu(\psi)$$

converges locally uniformly to

$$f(z)=\int_B\int_\Gamma \frac{1}{(1-\bar{x}\,\psi(z))^\beta}d\nu(x)d\mu(\psi).$$

Let $\eta_n(\psi)=\sum_{k=1}^{n}\nu_k\mu(\psi)$ then,

$$\int_B g_n(\psi)d\mu(\psi)=\int_B\frac{1}{(1-\psi)^\beta}d\eta_n(\psi),$$

where $\|\eta_n\| \le \|\,\|\nu\|\,\mu\|$ for all n. Hence by compactness, there exists a measure σ, such that,

$$f(z)=\int_B\frac{1}{(1-\psi(z))^\beta}d\sigma(\psi), \tag{2.5}$$

which shows that $f\in A_\beta$. Furthermore, $\|\sigma\| \le \|\nu\|\,\|\mu\|$. Consequently,

$$\|f\|_{A_\beta} \le \|\sigma\| \le \|\mu\|\;\|\nu\| \qquad \text{for all } \mu$$

however since μ and ν are arbitrary measures that give (2.1) and (2.2) then,

$$\|f\|_{A_\beta} \le \inf\{\|\mu\|\}\inf\{\|\nu\|\}$$

Hence by (2.3)

$$\|f\|_{A_\beta} \le \|f\|_{A_\alpha}$$

(ii) Now let $f\in A_0$. We want to show that $f\in A_\alpha$ for any $\alpha > 0$. By definition,

$$f(z)=\int_B \log\frac{1}{1-\phi(z)}d\mu(\phi)+f(0) \tag{2.6}$$

Since $\log\frac{1}{1-z}\in F_0 \subset F_\alpha$ (see Hibschweiler and MacGregor, 1989), then

$$\log\frac{1}{1-z}=\int_\Gamma\frac{1}{(1-\bar{x}z)^\alpha}d\nu(x)$$

where ν depends only on α. Hence (2.6) becomes

$$f(z)=\int_B\int_\Gamma\frac{1}{(1-\bar{x}\phi(z))^\alpha}\quad d\nu(x)d\mu(\phi)+f(0) \tag{2.7}$$

where the integral in (2.7) looks exactly like the one in (2.3) with α replacing β and hence using an argument similar to the one in (i) we will get that

$$f(z)=\int_B\int_\Gamma \frac{1}{(1-\psi(z))^\alpha}d\sigma(\psi)+f(0) \tag{2.8}$$

which shows that $f\in A_\alpha$. Furthermore $\|\sigma\|\leq\|\nu\|\,\|\mu\|$, hence

$$\|f\|_{A_\alpha}\leq\inf\{\|\mu\|\}+|f(0)|=\|f\|_{A_0} \tag{2.9}$$

Characterization of A_0

It is known (Garnett, 1980, p248) that a function $\phi\in$ BMO if and only if there exists functions ϕ_1 and ϕ_2 in L^∞ such that

$$\phi=\phi_1+\tilde{\phi}_2+\alpha$$

where both $\|\phi_1\|_\infty$ and $\|\phi_2\|$ are less than $C\,\|\phi\|_*$, C is a constant and $\|\ \|_*$ is the classical BMO norm (Garnett, 1980, p248).

Consequently $f\in$ BMOA if and only if three are analytic functions f_1 and f_2 such that

$$f=f_1+f_2+\alpha \tag{3.1}$$

where $\|\text{Re}\,f_1\|_\infty\leq C$ and $\|\text{Im}\,f_2\|_\infty\leq C$.

If we define on BMOA the norm

$$\|f\|^*=\inf\{\|\text{Re}\,f_1\|_\infty+\|\text{Im}\,f_2\|_\infty : f=f_1+f_2+\alpha\} \tag{3.2}$$

then by (Garnett, 1980, p248), the norms $\|f\|^*$ and $\|f\|_*$ are equivalent.

Now we have the following proposition which establishes a set equality between A_0 and BMOA.

THEOREM 2: $A_0=$ BMOA

Proof: Suppose that $f\in A_0$, then according to (1.1) and (1.2) there exists a measure $\mu\in N$ such that,

$$f(z)=\int_B \log\frac{1}{1-\phi(z)}d\mu(\phi)+f(0) \tag{3.3}$$

Assume without loss of generality, that μ is a probability measure. Then f is subordinate to $\log\frac{1}{1-z}+f(0)$ and consequently by (3.1), $f\in$ BMOA. The proof of the other inclusion follows from (3.1) and subordination.

THEOREM 3: The norms $\|.\|_*$ and $\|.\|_{A_0}$ are equivalent, namely there exists positive constant c_1 and c_2 such that

$$c_1\|f\|_*\leq\|f\|_{A_0}\leq c_2\|f\|_* \tag{3.4}$$

Proof: Suppose $f \in$ BMOA, then f can be decomposed as in (3.1). Let d_1 denote $\|\mathrm{Re}\, f_1\|_\infty$ and d_2 denote $\|\mathrm{Im}\, f_2\|_\infty$. Then

$$\left|\frac{\pi}{2d_1}\mathrm{Im}\, if_1(z)\right| \leq \frac{\pi}{2} \tag{3.5}$$

and

$$\left|\frac{\pi}{2d_2}\mathrm{Im}\, f_2(z)\right| \leq \frac{\pi}{2} \tag{3.6}$$

for all $z \in \Delta$. Consequently, by subordination

$$if_1(z) = \frac{2d_1}{\pi}\left(\log\frac{1}{1-\phi(z)} - \log\frac{1}{1+\phi(z)}\right) + if_1(0) \tag{3.7}$$

$$f_2(z) = \frac{2d_2}{\pi}\left(\log\frac{1}{1-\psi(z)} - \log\frac{1}{1+\psi(z)}\right) + f_2(0) \tag{3.8}$$

for all $z \in \Delta$ and where $\phi, \psi \in B$. Therefore

$$\| f \|_{A_0} \leq \frac{4}{\pi}(d_1 + d_2), \tag{3.9}$$

and hence

$$\| f \|_{A_0} \leq \frac{4}{\pi}\|f\|^* \leq c_2\|f\|_*, \tag{3.10}$$

which gives the right inequality in (3.3)

Next, we show the left inequality. Let us write f as in (1.2) and assume without loss of generality that μ is a positive measure. Then

$$|\mathrm{Im}\, f(z)| \leq c\|\mu\| \tag{3.11}$$

where $c > 1$. Thus

$$|\mathrm{Im}\, f(z)| \leq c\|f\|_{A_0} \tag{3.12}$$

and since

$$\|\mathrm{Im}\, f\|_\infty \leq \|\mathrm{Im}\, f\|_* \tag{3.13}$$

we have

$$\|f\|_* \leq k_1 \|\mathrm{Im}\, f\|_* \leq k_2 \| f \|_{A_0} = \frac{1}{c_1}\| f \|_{A_0}. \tag{3.14}$$

where $c_1 = \dfrac{1}{k_2}$ and the left inequality in (3.13) follows by (Garnett, 1980, p235) and this concludes the proof.

THEOREM 4: $\| z^n \|_{A_a} \leq k$ *for* $n \geq 1$

Proof: It is enough to show that $\| z^n \|_{A_0} \leq k$ for $n \geq 1$. Since we showed that $\| \cdot \|_*$ and $\| \cdot \|_{A_0}$ are equivalent, let us approximate $\|z^n\|_*$. It is known from ([2], p240) that

$$\| g \|_{*}^{2} \approx \sup_{\psi} \iint_{\Lambda} | \nabla g |^{2} \left(1 - | z |^{2}\right) | \Psi'(z) | \, dA \tag{3.15}$$

where $\psi(z) = \dfrac{z + z_0}{1 + z_0 z}$ is a Möbius transformation. Replace g in (3.15) by z^n and $|z|$ by r to get,

$$\begin{aligned}
I &= \iint_{\Lambda} \left|\nabla(z^n)\right|^2 \left(1 - r^2\right) | \psi'(z) | \, dA \\
&\leq \iint_{\Lambda} n^2 r^{2n-2} \left(1 - r^2\right) | \psi'(z) | \, dA \\
&\leq \int_0^{2\pi} \int_0^1 n^2 r^{2n-2} \left(1 - r^2\right) | \psi'(z) | \, dr d\theta \\
&\leq 2\pi n^2 \int_0^1 r^{2n-2} (1 - r) dr, \quad \text{because} \int_0^{2\pi} | \psi'(z) | \, d\theta \leq 2\pi \\
&\leq 2\pi n^2 \int_0^1 \left(r^{2n-2} - r^{2n}\right) dr = \frac{4\pi n^2}{\left(4n^2 - 1\right)} \leq \frac{4\pi}{3}
\end{aligned} \tag{3.16}$$

which gives us the desired result and completes the proof.

The following theorem is a direct consequence of Proposition 4.

THEOREM 5: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic and if $\sum_{n=1}^{\infty} | a_n | < \infty$ then $f \in A_\alpha$ for all $\alpha \geq 0$.

Proof. It is sufficient to prove that $f \in A_0$ since $A_0 \subset A_\alpha$. To show that $f \in A_0$ all we have to show is that the norm $\| f(z) \|_{A_0}$ is bounded.

$$\| f(z) \|_{A_0} = \left\| \sum_{n=0}^{\infty} a_n z^n \right\|_{A_0} \leq \sum_{n=0}^{\infty} |a_n| \; \| z^n \|_{A_0} \leq \frac{4\pi}{3} \sum_{n=0}^{\infty} |a_n| < \infty.$$

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