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Metrization of Weakly Developable Spaces

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مترية الفضاءات النامية الضعيفة

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خلاصة : نقدم في هذا البحث النظرية المترية للفضاءات النامية الضعيفة.

ABSTRACT: In this note, we present metrization of weak developability.

Keywords: Weakly Developable, Metrizable, Weak Base.

1. Introduction

Martin (1976) introduced the concept of weak developability in order to study the problem of the metrization of spaces with weak bases. A space X is weakly developable if and only if there is a sequence $\{G_n\}_{n\in N}$ of covers of X such that for each $x \in X$,$\{st(x,G_n): n \in N\}$ is a weak base at x . The sequence $\{G_n\}_{n\in N}$ is said to be a weak development for the space X . If each G_n consists of open sets, then $\{G_n\}_{n\in N}$ is a development for the space X and X is a developable space (Gruenhage, 1984).

The idea of weak base was introduced by Arhangel'skii (1966) in the study of symmetrizable spaces. It is more convenient to use the form of Siwiec, (1974) and Franklin, (1965).

A collection w of subsets of a space X is called a weak base for X provided that to each $x \in X$, there exists $w_x \subset w$ such that

1. Each member of w_x contains x .
2. For any two members W_1 and W_2 of w_x , there is a W_3 in w_x, such that $W_3 \subset W_1 \cap W_2$.
3. A subset F of X is closed if and only if for every point $x \notin F$, there exists a W in w_x such that $F \cap W = \phi$.

If to each $x \in X$ we assign a collection w_x of supersets of $\{x\}$ such that $W = \cup\{w_x : x \in X\}$ is a weak base by virtue of the collections w_x, i.e., the collections w_x satisfy conditions (1), (2) and (3) of the preceding paragraph, then we say that the collection w_x is a local weak base at x for each $x \in X$. It is easy to show that a subset O of a space X with local weak bases $\{w_x : x \in X\}$ is open if and only if for each $x \in O$, there is a member W of the local weak base w_x of $x \in X$ with $W \subset O$.

In this study, we prove a metrization theorem for weakly developable spaces. We assume throughout this note that all spaces are T_0. A topological space is a T_0-space if, and only if, for each pair x and y of distint points, there is a nighborhood of one point to which the other does not belong. Also, we let N denote the set of all positive integers. For a collection G of subsets of a space X, we define $st(x,G)=\cup\{g : x\in g\in G\}$ and $st^2(x,G)=\cup\{st(y,G) : y\in st(x,G)\}$.

2. Main Results

2.1 lemma 1

Let $\{G_n\}_{n\in N}$ be a weak development of a space X. Then for every compact subset K of X and any sequence $\langle y_n : n\in N\rangle$ of points in K, there is a point y in X and a subsequence $\langle y_n : i\in N\rangle$ of the sequences $\langle y_n : n\in N\rangle$ which converges to y.

Proof. Let K be a compact subset of X and let $\langle y_n : n\in N\rangle$ be any sequence of points in K. Suppose there is no subsequence of $\langle y_n : n\in N\rangle$ which converges to a point of $X-\{y_n : n\in N\}$. Then, we note that $F=\{y_n : n\in N\}$ is a closed subset of X. For if it is not true then for some point $y\in X-F$ we shall have $y_{n_1}\in st(y,G_i)\cap F$ for each $i\in N$. This will imply that the subsequence $\langle y_n : i\in N\rangle$ will converge to y, which will contradict our assumption. Therefore, F is closed.

Define $F_n=\{y_i : i\geq n\}$ for each $n\in N$. Similarly, one can show that F_n is closed for each $n\in N$. Consider the open cover $\{X-F_n : n\in N\}$ of K in X. It is easy to see that it does not contain a finite subcover of K which contradicts the fact that K is compact. Hence, the sequences $\langle y_n : n\in N\rangle$ have a convergent subsequence.

2.2 lemma 2

Let $\{G_n\}_{n\in N}$ be a weak development of a space X, which satisfies the following condition: For any closed subset F of X and any point $y\in X-F$, there is an $n\in N$ such that $st(y,G_n)\cap st(F,G_n)=\phi$. Then every compact subset of X is closed.

Proof. Let K be a compact subset of X. Suppose K is not closed. Then there is a point $y\in X-K$ such that $st(y,G_i)\cap K\neq\phi$ for all $i\in N$. Thus for each $i\in N$, let $y_i\in st(y,G_i)\cap K$. Hence, the sequence $\langle y_i : i\in N\rangle$ converges to y. Put $F=\{y_i : i\in N\}\cup\{y\}$. We claim that F is a closed subset of X. For if not, then there is a point $z\in X-F$ such that $st(z,G_i)\cap F\neq\phi$ for all $i\in N$. Without loss of generality, let $y_i\in st(z,G_i)\cap F$ for all $i\in N$. This is not possible since a weakly developable space is T_1 and hence by the hypothesis there is an $n\in N$ such that $st(z,G_n)\cap st(y,G_n)=\phi$. Define $F_n=\{y_i : i\geq n\}\cup\{y\}$ for each $n\in N$. Clearly, F_n is closed for each $n\in N$. Therefore, $\{X-F_n : n\in N\}$ is an open cover of K with no finite subcover of K giving a contradiction.

2.3 Theorem

The following are equivalent for a space X.

1. The space X is metrizable.
2. The space X has a weak development $\{G_n\}_{n\in N}$ such that for any closed subset F of X and any point $x\in X-F$, there is an $i\in N$ such that $st(y,G_i)\cap st(F,G_i)=\phi$

3. The space X has a weak development $\{G_n\}_{n\in N}$ such that if $A \subset V$, where A is compact and V is open, then $st\left(A, G_n\right) \subset V$ for some n.

4. The space X has a weak development $\{G_n\}_{n\in N}$ such that if $x \in V$, where V is open, then there exists a neighborhood U of x *and* $n \in V$, for which $st\left(U, G_n\right) \subset V$.

5. The space X has a weak development $\{G_n\}_{n\in N}$ such that if $x \in V$ is open, then there exists an $n \in N$ for which $st^2\left(x, G_n\right) \subset V$.

Proof. The implication $1 \Rightarrow 2$ is clear. The implications $3 \Rightarrow 4$, $4 \Rightarrow 5$ and $5 \Rightarrow 1$ are proved in Martin (1976, Theorem 2.5 and Theorem 2.6). To prove $2 \Rightarrow 3$. Let A be any compact subset of X and let V be an open subset of X containing A. Suppose that $st\left(A, G_i\right) \cap \left(X - U\right) \neq \phi$. Let $x \in st\left(A, G_i\right) \cap \left(X - V\right)$. Hence, by Lemmas 2.1 and 2.2 there is a subsequence $\left\langle x_{i_k} : k \in \mathrm{N}\right\rangle$ of the sequence $\left\langle x_i : i \in \mathrm{N}\right\rangle$, which converges to a point y in A. Now, by hypothesis there is a $j \in \mathrm{N}$ such that $st\left(y, G_i\right) \cap st\left(\left(X - V\right), G_i\right) = \phi$. This leads to a contradiction.

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