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On the Existence of Evolutionary Learning Equilibriums

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دراسة عن وجود توازنات تعليمية تطورية

مسعود العلم تشاودري

ملخص: إن نظريات النقطة الثابتة المبنية على أساس وجود توازن تنافسي والتي تشرح النظرية الاقتصادية في لب الاقتصاد، يمكن أن تعمل فقط على مجموعات محددة ومغلقة وذات منظورات أو تصورات معكوسة. لكن هذه الشروط نادراً ما تكون موجودة في العالم الوقعي للتعقيدات والأوضاع الاقتصادية والمالية المضطربة. إن فئة المجموعات التعلمية التي تبينها الحقول المستمرة لسلوكيات التفاعل والتكامل والتطور التي تسببها المفاضلات الديناميكية على المستويات الفردية والمؤسساتية والاجتماعية لا يمكن أن تلبي احتياجات فرضية الأوضاع المغلقة المحددة المعكوسة. لذا فالمجموعات التعلمية والعلاقات متعددة النظم ما بين الأزمان، التي توضحها التكاملات والمساهمات الواسعة ما بين المتغيرات والكيانات والتطور بفعل التعلم، لها توازنات تطورية .هكذا دراسة تتطلب طريقة منهجية جديدة.
هذه الورقة تصوغ هذه المنهجية للتوازنات التطورية في الفضاءات التعلمية ، وباختصار، تبين شمولية التوازنات التعلمية في كل التراكيب الرياضية .وبشكل محدد تبين الورقة الاعتماد المتبادل متداخل النظم ما بين التنمية المستدامة والأخلاق والاقتصاد في الفهم الخاص أو المحدد للمجال التعليمي.

ABSTRACT: The usual kinds of Fixed-Point Theorems formalized on the existence of competitive equilibrium that explain much of economic theory at the core of economics can operate only on bounded and closed sets with convex mappings. But these conditions are hardly true of the real world of economic and financial complexities and perturbations. The category of learning sets explained by continuous fields of interactive, integrative and evolutionary behaviour caused by dynamic preferences at the individual and institutional and social levels cannot maintain the assumption of closed, bounded and convex sets. Thus learning sets and multi-system inter-temporal relations explained by pervasive complementarities and participation between variables and entities, and evolution by learning, have evolutionary equilibriums. Such a study requires a new methodological approach. This paper formalizes such a methodology for evolutionary equilibriums in learning spaces. It briefly points out the universality of learning equilibriums in all mathematical structures. For a particular case though, the inter-systemic interdependence between sustainable development and ethics and economics in the specific understanding of learning domain is pointed out.

KEYWORDS: Economic theory; Mathematical economics; Evolutionary economics; Economics and epistemology.

1. Background: a brief review of the literature

The theme of the existence of equilibrium in mathematical economics used to be a hot pursuit of researchers in the sixties and seventies, earlier and later on (Brouwer, 1910; Chichilnisky, 1993; Kakutani, 1941; Debreu, 1959, 1989; Uzawa, 1964; Quirk and Saposnik, 1968; Morishima, 1964; Nikaido, 1989). This momentum has declined in recent years. But retrospectively, the steady-state and time-dependent optimal movements (Pontryagin *et al.* 1965; Intrilligator, 1971) of globally stable equilibriums have failed to produce the realism of a vastly unstable world of economics, finance and related areas that we have inherited. The real world of economics and finance is based on complexities and perturbations (Bertuglia and Vaio, 2005; Casti, 1990) rather than on any visage of steady-state equilibrium or even time-dependent optimal states of equilibrium. There are deep epistemological issues in these theoretical questions. These were noted by Shackle (1971). Choudhury (2006) has formulated a description of a non-steady *qua* time-dependent equilibrium socioeconomic space and characterized it by the super-encompassing knowledge-induced learning states of the economic and finance space. This paper formalizes the existence of equilibrium in learning systems of state-variables and policy-variables under conditions of endogenous preferences and inter-systemic variable-vide relationships. Indeed, without equilibrium, the predictability of a system even in the neighbourhood of occurrence of events, what I refer to as 'near' points of occurrence of an event (Choudhury, 2010), cannot take place. We would then be led into a chaotic world without order. This would not be to the social benefit of stabilizing the world-system as it prevails today, and thereby attaining well governed socioeconomic reconstructions out of a fallen world (Krugman, 1996). Yet the nature of such equilibriums, as they belong to a learning domain formed by complexities and perturbation, have been well studied by Grandmont (1989), Shell (1989) and Thurow (1996).

2. Objective

The objective of this paper is firstly to formulate the important theme of existence of endogenously knowledge-induced equilibriums. Secondly, the problem of existence of such overarching domains of equilibriums induced by learning processes within (intra-) and across (inter-) systems and of their state-variables is formalized. A simple problem on the existence of the resulting evolutionary learning forms of equilibrium intra- and inter- systems is proved. Although the substance of the paper overarches across the entire socio-scientific field, the topics here are narrowed down by invoking specific intra- and inter- system problems in economics and finance. Attention is given to the topic of endogenous development sustainability. The subject matter taken up as a whole is in the area of neural economics and complexity.

3. Explaining evolutionary learning equilibriums

We define evolutionary learning equilibrium as intra- and inter- systemic in respect of their intra- and inter-systemic variables and their relationships defined by the following expressions:

3.1 Intra-systemic evolutionary learning equilibriums

There exists a sequence of 'temporary' equilibriums, E_s^*, in a given system denoted by s. This system is denoted by $E_s^* = \{e_i^*\}_s$, where i = 1, 2, … within system s, denotes the number of events. Every event is denoted by $\{e_i^*\}_s$ in $E_s = \{e_i\}_s$. In an applicable domain of intra-system **I**nteractive, **I**ntegrative (**II**)-learning processes, the following is true: (1) $\left\{E_s - E_s^*\right\} < \left\{\varepsilon_s\right\}$, a small positive value conforming to corresponding numbered events $\left\{e_i^*\right\}_s$, i = 1, 2, … , for a given system numbered as s and events in E_s and E_s^* belonging to s.

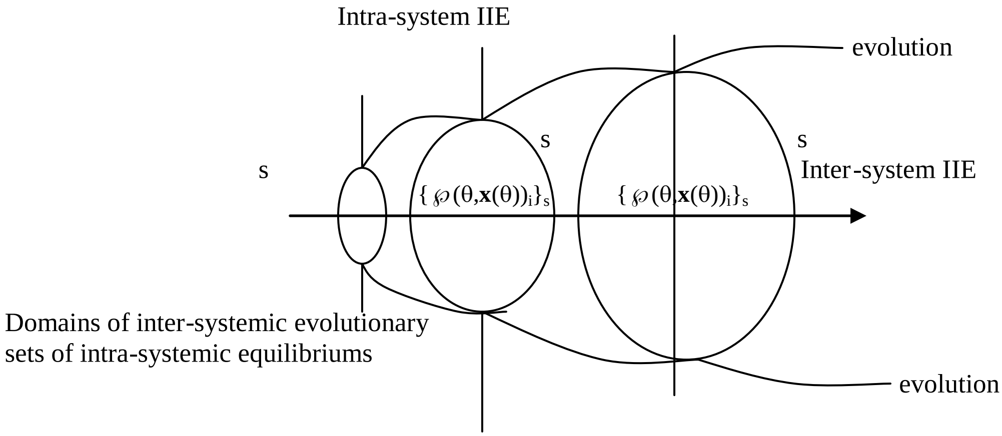

Figure 1. Intra-system and inter-system domains of evolutionary equilibriums

Elements of $\left\{\varepsilon_s\right\}$ can assume real continuous values or integer values when discrete depending upon the kind of event space E_s s. (2) But *none of the* ε_s *is an indefinitely small positive value*. This property of the equilibrium event allows for evolution at any state of knowledge *before* its exact convergence is attained. The same kind of institutional and social state is found in many instances.[1] Convergence in the $\left\{\varepsilon_s\right\}$-set for

[1] The same rule complies with consensus in Robert's Rule of democratic procedure. It is also the basis of the principle of 'irrelevant preferences' in social choice theory (Arrow, 1951); enforcement of ethical preferences (Hammond, 1989); and terminating of a Rawlsian continuous mini-max game by an external preceptor (Wolfe, 1977). Also sudden great departures from the normal sciences are explained by scientific revolutions as of Kuhn (1970), and complexity in historical change of the longue duree (Braudel, 1980).

[2] The usual definition of continuity around a point θ* of θ-values affecting an event E_s (say state variable) is this: $|E_s(\theta) - E_s(\theta^*)| < \delta$, for $|\theta - \theta^*| < \varepsilon_s$, for (δ, ε_s) as indefinitely small positive values. But in the case of an evolutionary equilibrium described by knowledge-trajectories over *expanding* knowledge-values {θ} across evolutionary events that are simultaneously intra-system and inter-systems, we have the result: $|E_s(\theta) - E_s(\theta^*)| = |(E_s(\theta) - E_{s1}(\theta_1)) + (E_s(\theta_1) - E_{s2}(\theta_2)) + \ldots (E_s(\theta_n) - E_s(\theta^*))| < \delta_1 + \delta_2 + \ldots + \delta_n$. This aggregate value is not an indefinitely small number by virtue of addition of finitely many positive small values to allow for finite time consensus without unanimity in agreement. Along with this kind of a non-convergent learning process exactly to θ* (non-unanimity though consensus), we then also have, $|\theta - \theta^*| = |(\theta - \theta_1) + (\theta - \theta_2) + \ldots + (\theta - \theta_n)| < \varepsilon_{s1} + \varepsilon_{s2} + \ldots + \varepsilon_{sn} = \{\varepsilon_s\}$, which is not an indefinitely small number by virtue of the addition of small values that are not indefinitely small along *expanding* θ-values. Hence $\{\varepsilon_s\}$ is a set of small positive numbers but these are not indefinitely small. Thereby the same property is shared by $\{E_s(\theta)\}$. All these mean that there is no steady-state equilibrium in sets of evolutionary equilibriums. No exact equilibrium is reached. Evolutionary equilibrium is only approximated to before next evolution takes off.

[3] The negative value of the time derivative means the convergence of E_s to E_s^* as learning continues, that is with $\theta \to \theta^*$. An example is the case of equilibrium by cobweb tatonement adjustment. The excess demand progressively diminishes with rounds of adjustment. Such adjustments take place over time, although time is implied in the static form of the equilibrium tatonement process in excess demand (Henderson and Quandt, 1971).

[4] The following expression is a misnomer: $W = f(x(\theta),t)$, for t is exogenous in this relation and does effect f(.) and its variables endogenously. Rather, we would write the above functional relationship as, $W_t = f(x_t(\theta_t))$, for data observed and simulated at time t. Here values of t could be discrete or continuous. In the latter case, 't' takes values on the real line $\Re$ or real N-multidimensional space $\Re^N$.

[5] A non-compact or equivalently non-fixed point type topological set is an open, unbounded and not necessarily convex set in $\Re^N$. For technical details see Friedman (1982) and Debreu (1959).

$E_s \to E_s^*$ is continuous but only to the neighbourhood of E_s^*, which is not indefinitely small as in the standard definition of continuity.[2]

Because the $\{E_s\}$ are knowledge-induced events caused by learning processes, such events are characterized by their variables denoted by $\{(\theta, x(\theta))_i\}_s$ {(θ,**x**(θ))i}s. This means that there is a limiting learning parameter θ intra-system that characterizes (endogenizes) the state-variables $x(\theta)$ of every event *within* that system. Because of the learning nature of $\{(\theta, x(\theta))_i\}_s$ these occur out of consensus in every round of the interactive and integrative (**II**) learning process within each given system of variables and entities in the learning system.

Therefore, every vector of events is defined by $E_s = \{(\theta, x(\theta))_i\}_s$. Consequently, the above characterization of learning equilibriums is taken up individually *within* a system, but attained by interrelations between the knowledge-flow variable $\{\theta\}$ and the knowledge-induced vector of state-variables $\{x(\theta)\}$ in that system. In this case, the evolutionary equilibrium phenomenon is realized intra-system.

This kind of intra-system movements occurring across domains of evolutionary equilibriums can be depicted in Figure 1. But Figure 1 also explains evolutionary inter-system equilibriums emerging from the intra-systemic ones. We now explain this second case of evolutionary equilibriums.

Every preference map in the learning space, which affects the event space, is formed by consciousness formed by continuous learning processes. Consciousness is thus embedded in a socio-scientific state of knowledge-induction. Thereby, consciousness that is invoked by dynamic endogenous preferences in the **II**-learning process is defined by the preference set, $\{\wp(\theta, x(\theta))_i\}_s$. Note how the theme of consciousness assumes a great scientific meaning in our times, as it has been during the classical times marking great scientific contributions (Heisenberg, 1958; Kafatos and Nadeau, 1990).

We now summarize all the above-mentioned properties of equilibrium *within* an s-system in terms of its given preferences and events: The intra-systemic event-specific learning equilibriums are induced by the episteme of unity of knowledge. Such an episteme characterizes the nature of convergence by complementarities and participation between entities and their representative variables along the **II**-learning process *within* a given s-system.

Next we examine the second characteristic of Figure 1 on evolutionary inter-systemic equilibriums. The inter-systemic evolutionary equilibriums emerge from the learning processes of the intra-systemic equilibriums.

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[6] Define metric by a distance function d(x,y) that is explainable over any set described by the sequence of variables say {x,y}, for which there exist well-defined functions f(x) and f(y), respectively, such that the following postulates hold: (1) d(x,y) = 0 if and only if x = y; (2) d(x,y) = d(y,x); (3) dx,z) ≤ d(x,y) + d(y,z), for z ∈ (x,y}. All variables are real-valued. Conformably, if the metric is a topology where order and mappings are preserved over {x,y}, then, f(x), f(y) and f(z) will maintain the above-mentioned properties for themselves and for the sequence of variables.Clearly, the above definition of the metric is invalidated for learning spaces such as {θ, x(θ), y(θ), z(θ)}. This is also true of the functional denoted by W(θ, x(θ), y(θ), z(θ)) defined on {θ, x(θ), y(θ), z(θ)}; $\theta \in \Re^N$. The following properties of the learning space causes the invalidation of the metric-definition: {x(θ)} → {y(θ)} /⇒ {y(θ)} → {x(θ)} by an identical mapping (like I); the same is true of {z(θ)}. Likewise, dW/dθ > 0, without there being a value of η > 0, such that $/W – W_o/ < \eta$, for 0 < d(x(θ),y(θ)) < ε.

[7] Brouwer's Fixed Point Theorem, which is the genesis of all subsequent fixed-point theorems states: If f is a continuous function from a closed bounded convex set of Euclidean space C into itself, then there exists x*∈ C, such that f(x*) = x*.

[8] The idea of the good things of life is exemplified by Rawls' (1971) primaries and in his definition of the Difference Principle for the wellbeing of the most underprivileged.

3.2 Definition of evolutionary learning equilibrium: from intra-system to inter-system

The *evolutionary learning equilibriums* across system interrelations between state-variables is denoted by indexed $\{\theta, x(\theta)\}_i$, i =1,2, (includes policy-variables in the **x**(θ)-vector) that appear *within* an s-system. This dynamics is now defined as follows:

For every member element i = 1,2,… *within* an s-system, there exist $E_s^* = \left\{\left(\theta^*, x^*\left(\theta^*\right)\right)_i\right\}_s$. The asterisked symbols denote limiting values where evolution across systems occurs. Now for every i-tuple, evolutionary system-ensemble is defined by, $E_s = \left\{\left(\theta, x(\theta)\right)_i\right\}_s, \left\{E_s - E_s^*\right\} < \left\{\varepsilon_s\right\}$, for each value in $\left\{\varepsilon_s\right\} > 0$, *but not indefinitely small value* in the neighbourhood of $\left\{E_s - E_s^*\right\}$.

The intra-systemic events in s-system so taking place along the **II**-learning process are driven by consciousness that is represented in preference maps $\left\{\wp\left(\theta, x(\theta)\right)_i\right\}_s$ as these are induced by the epistemic unity of knowledge θ. We write such events as $\left\{\theta, x(\theta)_i\right\}_s$ occurring in recursive rounds of **II**-learning intra-s-system but emerging across systems at the limiting values shown by asterisk.

Preference maps can only be implied as a matter of inducing and inculcating consciousness. Yet they cannot be measured in the commensurate sense. Therefore, the conceptual-empirical implication here is to subsume the entire definition and measurement of equilibrium intra-s-system with the primordial role of knowledge-flows derived from the episteme of unity of knowledge in the first place. Knowledge-flows as observable variables reflect levels of attained consciousness. Knowledge-flows alone can be commensurably ordinalized through social discourse and measurement in the conceptual-empirical problems under investigation.

The intra-s-systemic evolutionary learning dynamics across sequences of evolutionary equilibriums is now defined by the complete **I**nteractive-**I**ntegrative-**E**volutionary (**IIE**)-learning processes intra-s-system by adding the following property of evolutionary equilibriums. Evolution across systems as indicated in Figure 1 has now taken place.

Define $\Delta \mathrm{E}_s = \left\{\mathrm{E}_s\text{-}\mathrm{E}_s^*\right\}$, with $\mathrm{E}_{i,s,n} = \left\{\left(\theta_n, \mathrm{x}_n, (\theta_n)\right)_i\right\}_s$, n denoting new evolutionary i-designated values of events when such values span across systems. $\Delta \mathrm{E}_s$ are determined along the evolutionary learning processes by the predetermined limiting $\left\{\left(\theta^*, x^*\left(\theta^*\right)\right)_i\right\}_s$-values intra-system for given s and i = 1,2,… events *within* system s. We therefore characterize such evolutionary equilibriums across i-variables *within* a given s-system now spanned across n-evolutionary systems as follows:

$$\begin{aligned} \frac{d\Delta E_s}{d\theta^*} &= \sum\nolimits_{i(s)} \left(\frac{\partial \Delta E_s}{\partial \theta_n}\right) \cdot \left(\frac{d\theta_n}{d\theta^*}\right) = \sum\nolimits_{i(s)} \left(\frac{\partial \Delta E_s}{\partial \theta_n}\right) \cdot \sum\nolimits_n \left(\frac{\partial \theta_n}{\partial x^*\left(\theta^*\right)}\right) \cdot \left(\frac{dx^*\left(\theta^*\right)}{d\theta^*}\right) \\ &= \sum\nolimits_{i(s)} \sum\nolimits_n \left[\left(\frac{\partial \Delta E_s}{\partial \theta_n}\right) \cdot \left(\frac{\partial \theta_n}{\partial x^*\left(\theta^*\right)}\right) \cdot \left(\frac{\left(dx^*\left(\theta^*\right)\right)}{d\theta^*}\right)\right] < 0 \end{aligned} \tag{1}$$

The governing property for determining the limiting convergence out of learning gives the negative sign of $d\Delta E_s/d\theta^*$ *identically* for each i-variable in s-system. We denote this simultaneous occurrence of variables in numbered system (s) by $i(s)$.

Thus, $(\partial\Delta E_s/\partial\theta_n) < 0$ for $i(s) = 1,2,\ldots$ values specific to s-system study; n denotes the evolutionary states of the new iterative E-values that become convergent to E^*-values within $i(s)$. Liapunov (La Salle & Lefschetz, 1961) has formalized a dynamic condition of stable time-dependent equilibrium state to exist. The difference of the above-mentioned dynamic knowledge-induced evolutionary equilibrium from Liapunov's condition is that there is no need for a neighbourhood of an indefinitely small positive size denoted by $\{\varepsilon_{i(s),n}\}$.

Expression (1) is thus indicative of progressive evolution of n-new inter-system values of $\{\theta,\mathbf{x}(\theta)\}_{i(s)}$, n being across phases of **IIE**-learning processes, hence across systems (inter-systems). Such a path of evolutionary **IIE**-learning processes *within* s-system marks the path-dependency creative 'history' of the learning trajectory of $\{(\theta_n^*,\mathbf{x}_n^*(\theta_n^*))_i\}_s$-values. Expression (1) explains the conceptual part of the conceptio-empirical phenomenological model of learning processes in unity of knowledge. It need not be empirically computed. It is epistemologically invoked to establish consistency and sustainability of intra- and inter- system-learning *res extensa*.

Thus, expression (1) characterizes the nature of evolutionary equilibrium *within* the s-system across $i(s)$-state variables (including policy variables) pertaining to events, and spanned across n-iterative learning processes, hence inter-systems. This kind of bundles of evolutionary equilibriums determined by **IIE**-learning processes establishes the sustainable 'history' of unity of knowledge intra- and inter- systems.

4. Evolutionary equilibrium across *intertemporal* knowledge-induced paths of the IIE-learning processes: learning dynamics in the knowledge-time-space dimension

The intertemporal description of evolutionary learning equilibriums intra- s-system is obtained by differentiating (1) in respect of time. Thus,[3]

$$\left(\frac{d}{dt}\right)\left(\frac{d\Delta E_s}{d\theta^*}\right) < 0 \tag{2}$$

Expression (2) conveys two important meanings. Firstly, over time the nature of *intra*-systemic equilibrium is retained. The growth of knowledge intensifies the occurrence of evolutionary learning equilibriums. Hence sustainability of the endogenous circular causation relations of unity of knowledge is increasingly established along the intertemporal **IIE**-learning processes. Secondly, the changes of $\{\theta, x(\theta)\}_t$-values over time t across $i(s)$ and n number of phases *intra-* s-system evolving into inter-systems means that time simply *reads and records* the state of the $\{\theta,\mathbf{x}(\theta)\}_t$-values intertemporally. Note that we have dropped the subscripts in the variables to ease the symbolization. Such intertemporal recording of the $\{\theta, x(\theta)\}_t$-values does not mean that time has anything to do with primal causation of change in these values.[4] Only knowledge causes change. Time records change that is caused by knowledge induction of the intra- and inter- systemic variables.

5. Summarizing the analytics of the generalized form of evolutionary equilibriums

We can now summarize the evolutionary learning equilibrium in terms of the following operator, $(d/d\theta_N)$ (Choudhury and Zaman, 2006):

$$(\mathrm{d}/\mathrm{d}\theta_{\mathrm{N}})\left[\cup_{\mathrm{N}=t} \cap_{\mathrm{i,s}} \mathrm{W}(\mathrm{E}_{\mathrm{i,s}})[\theta_{\mathrm{N}}]\right] > 0, \mathrm{I} = 1,2,\ldots; s = 1,2,\ldots; N = 1,2,\ldots. \tag{3}$$

Underlying the evaluation of W(.) there is the system of circular causation that reflects organic unity of knowledge according to the principle of pervasive complementarities between variables and entities.

6. Fixed point theorem in non-compact topology[5]

Because of the openness of the learning domain of $\{\varepsilon(\theta)\}$, both for intra-systemic and inter-systemic changes over time, the equilibrium points on the trajectory of 'history' (i.e. such knowledge-induced changes recorded and read over time) belong to open and moving sets. Such sets are thereby of the type having non-fixed equilibrium points (Maddox, 1970)[6]. Hence there must be a revision of the Fixed Point Theorem in such a case of evolutionary learning equilibriums across non-compact topological spaces. For a characterization of such non-compact learning sets as topology see Choudhury *et al.* (2006). We can now enunciate a theorem on evolutionary Fixed-Point Theorem[7] for the case of non- compact evolutionary learning equilibrium points:

6.1 Statement of theorem

Given all the properties of evolutionary learning equilibriums, there does not necessarily exist a diminishing metric in an evolutionary learning set, such that $\{\varepsilon(\theta)\} > 0$ has an indefinitely small value. Thus any such set will not necessarily have a limiting point in $\{E(.)\}$ {E(.)}. A temporary limiting point will thus be pushed outside any particular ε-neighborhood of a point in the continuously evolutionary path of learning. Since this condition of evolutionary learning equilibriums applies to all neighborhoods of $\{\theta, x(\theta)\}_t$ -variables over time t, there is no convergent limiting point in all of $\{E(.)\}$.

7. Towards proving the existence of evolutionary learning equilibriums

To prove the existence of the consequential evolutionary learning equilibriums, the objective criterion $W(E(\theta))$ needs to be evaluated by means of simulation of circular causation relations at the 'nearest' point of evaluation of the state of the targeted variables in accordance with the episteme of unity of knowledge.

The 'predictor' values of the variables and θ representing a wellbeing index are simulated by changes in the regression coefficients of the circular causation equations for attaining certain degrees of unity of knowledge between the variables. The resulting simulated learning equilibriums form ‘non-fixed point’ type of sub-sets of $E(.)$ -event spaces across intra- and inter- systemic **IIE**-learning processes over time.

The proof of this theorem is given in the appendix. The footnote here explains the underlying idea of knowledge-induced magna of learning metrics. It is interesting to point out here that the same idea of the knowledge-induced metric also applies to the physical geodesic (Wald, 1992) of the open-universe conjecture of Popper (1988) upon which the realm of consciousness abides. Thus the body of application of the formalism of evolutionary equilibriums that learn over 'non-fixed point' topological sets universally spans all socio-scientific domains.

8. What is the place of economics and finance in the E(.)-generalized system formalism?

The formalism thus far on evolutionary learning equilibrium is of a generalized type. It can be used as mathematical formalism in modeling every problem of the socio-scientific category. In order to particularize to the special case of economics, finance, and related disciplines, we note in the general case signified by (3), that a vast nexus of interconnected complementary (participatory) variables arise across various domains specific to the problems at hand.

8.1 Exemplar: sustainability in socioeconomic development by the learning paradigm

Some of the socioeconomic problems addressed by the evolutionary equilibrium paradigm are the following: The study of inter-temporal sustainable development can be done by means of dynamic coefficients input-output matrices of learning inter-sectorally linked (complementary). Here, sectoral outputs can be complemented together to explain the evolutionary dynamics of an inter-sectorally connected joint production function. The same kind of intertemporal and inter-sectoral output matrix can be deconstructed into the matrices of derived demand and supply of productive factors. Factor prices can thereby be simulated by considering the derived factor demand and supply functions with complementarities between the factor inputs and their relative prices in the presence of the learning impact of the knowledge-flow θ-values to construct complementarities. Indeed, intra- and inter- systemic complementarities are the sure mark of organic unity of knowledge in the 'joint production function'. The induction of $x(\theta)$-vector by θ-values across **IIE**-learning processes signifies the endogenous nature of ethics in the problem under study.

The fields of ethics and economics as endogenously embedded systems become a deep study of sustainability paradigm in development in learning systems. The sustainability concept in the realms of ethics and economics is now applied to the study of inter-systemic complementarities (systemic participatoriness) comprising economics, ethics (society), and development sustainability.

Learning models of unity of knowledge of the type presented here enable the study of interconnected learning systems with evolutionary learning equilibriums in them (Choudhury *et al.* 2007a ; Choudhury *et al.* 2007b). Examples are sustainability in development paradigms and interlinking of money with real economy by development financing instruments (Choudhury, 2009). Inter-linkages by complementarities in the $x(\theta)$-vector of variables in such studies are the result of ethico-economic formalism premised on the episteme of unity of knowledge.

9. Development sustainability in the simulated domains of evolutionary learning equilibriums

We define development sustainability here by simulated learning between appropriate technology (θ), θ-induced output vector $(x(\theta))$ and average cost of production $AC(\theta, x(\theta))$. Goulet (1995) has given a succinct treatment of ethics as an endogenous force in sustainable development.

In the learning model of sustainable development, $AC(.)$ will decline with the growth of $(\theta, x(\theta))$; and there will be a positive relationship between $\{\theta\}$ and $\{x(\theta)\}$. Thus in a simulated domain of learning processes there will be systems of positive relations in the $(\theta, x(\theta))$-tuples and negative relations between $\{AC(.)\}$ and each of the variables of the simulated $\{(\theta, x(\theta))\}$-tuples. Such simulated regions comprising the totality of $\{\theta, x(\theta), AC(\theta, x(\theta))\}$ are not necessarily convex. Hence the fixed-point theorem cannot be used to prove the existence of equilibrium in evolutionary learning regions of simulation caused primordially by the effect of knowledge-flows $\{\theta\}$-values. Such relations in the simulated $\{\theta, x(\theta), AC(\theta, x(\theta))\}$-region bring out the property of pervasive complementarities between the good things of life[8] in a dynamic sense.

Simple technical details of the above example are examined below. The results of evolutionary equilibrium in simulated regions of sustainable development are shown in Figure 2.

In any given regime of development, hence in a given i(s)-stage of development with real GDP (x(θ)) and technology variable θ let,

$$AC\left(\theta, x\left(\theta\right)\right) = A + B.x\left(\theta\right) + C.\theta^{2} \tag{4}$$

Note that the coefficients A, B, C are themselves functions of θ because of the continuous shifts in the θ-induced simulated values $\left\{\theta, x\left(\theta\right), AC\left(\theta, x\left(\theta\right)\right)\right\}$. The properties of expression (4) in any intra-systemic $\left\{\theta, x\left(\theta\right), AC\left(\theta, x\left(\theta\right)\right)\right\}$-region followed by evolution for development sustainability inter-systems are worked out below

$$\frac{dAC}{d\theta} < 0, \Rightarrow \frac{dA}{d\theta} + B.\left(\frac{dx}{d\theta}\right) + x\left(\frac{dB}{d\theta}\right) + 2C.\theta + \left(\frac{dC}{d\theta}\right).\theta^{2} < 0 \tag{5}$$

By the effect of θ-learning on the other variables and parameters we obtain the following results:
$dx/d\theta > 0$; $dA/d\theta < 0$; $dB/d\theta < 0$; $dC/d\theta < 0$. There are no specific signs of A, B, C. But positive learning effect will make these shift-parameter enhancing wellbeing in the sustainable development model. Resulting from expression (5) we note,

$$B\left(\frac{dx}{d\theta}\right) + 2.C.\theta < \lambda\left(\theta\right) \text{ where } \lambda\left(\theta\right) = -\left[\frac{dA}{d\theta} + x\left(\frac{dB}{d\theta}\right) + \theta^{2}\left(\frac{dC}{d\theta}\right)\right] > 0$$

Expression (5) can be worked out with the help of the ancillary properties of learning effects as shown above, to show:

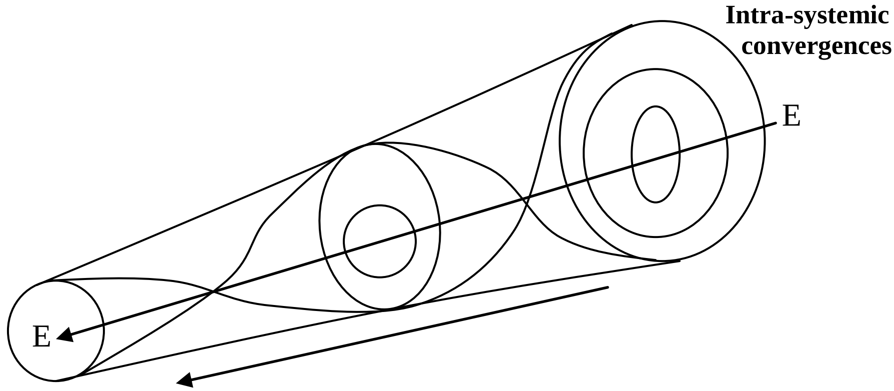
$$\frac{dx}{d\theta} = \left[\frac{\left(\lambda\left(\theta\right) - 2C.\theta\right)}{B\left(\theta\right)}\right] = \xi\left(\theta\right) > 0 \tag{6}$$

Thereby, $x\left(\theta\right) = \xi\left(\theta\right) * \theta$, for $x = 0$, for $\theta = 0$ of the learning type [see footnote 7]. If the variables of $\left\{\theta, x\left(\theta\right), AC\left(\theta, x\left(\theta\right)\right)\right\}$-region are taken in the natural logarithmic form then,

$$x\left(\theta\right) = \theta.e^{\xi\left(\theta\right)} \tag{7}$$

Figure 2 shows all the results of expressions (4), (6)/(7).

According to endogenous development paradigm in this paper, the History EE is fully encapsulated in expression (3), which is limited subject to circular causation relations in $i\left(s\right)$ across {θ, **x**(θ), AC(θ,**x**(θ))}-region in $i\left(s\right)$ [see earlier notation].While EE is driven across systemic continuums by knowledge-flows and knowledge-induced state and policy variables $\left\{x\left(\theta\right)\right\}$, and thus by circular causation relations between the inter-regional variables that enable evaluation of the wellbeing function, W(.), as shown above, yet the measurement of such system-evolutionary variables is recorded at specific time-periods. The resulting paradigm of endogenous development is different from the idea of endogenous economic growth (Romer, 1986). This is due to the absence of circular causation in endogenous growth theory, which otherwise, is central to the endogenous θ-effect in evolutionary $i(s)$-inter-systems.



Inter-systemic evolution and convergence through intra-systemi Linkages of the specific convergence on the evolutionary phases: the case of evolutionary average cost of production that never reaches a minimum point due to the technological effect of evolutionary learning.

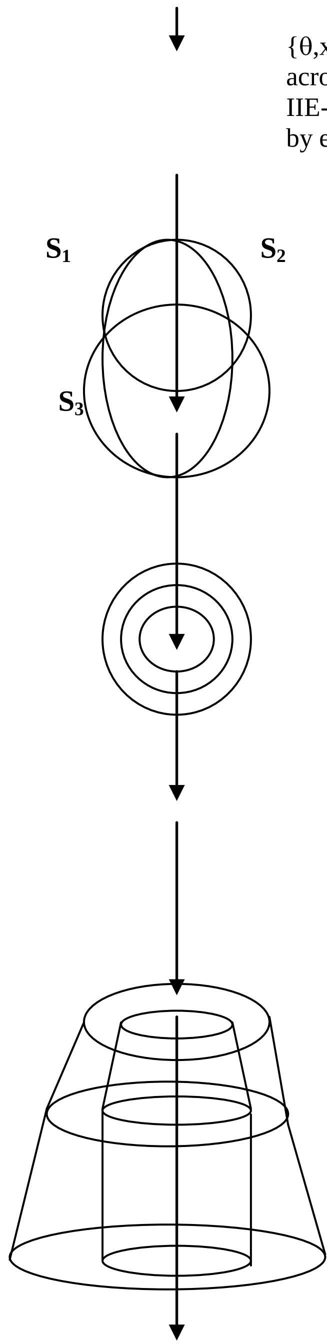
Figure 2. Development sustainability by endogenous technological induction.

Intra-inter-systemic convergence 'E' evolutionary equilibriums that describe what we refer to as evolutionary learning history. This is denoted along EE. Along EE the knowledge-time-space evolution is described by the trajectory of the evolutionary bundle caused by the continuous effect of θ-values, {θ, x(θ), AC(θ,x(θ))}-region in i(s) [see in text]. Inward movement of {θ, x(θ), AC(θ,x(θ))}-region denoted by the intra-inter-systemic gyration caused by learning affecting the continuously shifting curves under the primordial effect of θ-values, where shifts occur causing new evolutionary{θ, x(θ), AC(θ,x(θ))}-regions moving inwards. The origin E of evolutionary convergence is one of many such temporary evolutionary intra- and inter-systemic evolutionary fixed points (see Figure 1).

10. Conclusion: universalizing the evolutionary learning equilibriums

The sustainable **IIE**-learning system over knowledge-time-space dimension as explained by expression (2) can be generalized over any number of s-systems, with i-events and n-learning iterations of learning processes (hence inter-system evolutions). Consequently, multivariate and multidimensional $i(s)$-variables, intertemporally spanning any number of n-emergent IIE-learning processes, inter-systems induce experience in synergistic symbiosis in evolutionary learning. The global pictures of evolutionary learning equilibriums are now established as a sustainable trajectory of ‘History’ across knowledge-time-space dimension intra-systems and inter- systems. Such ensembles across history are explained by Hubner *et al.* (1985). See appendix 2 for a formal explanation of the specific application of evolutionary learning equilibriums to the theme of development sustainability.

$(d/d\theta_N)[\cup_{N=t}\cap_{i,s}W(E_{i,s})[\theta_N] > 0$, $i = 1,2,\ldots$; $s = 1,2,\ldots$; $n = 1,2,\ldots$.



$\{\theta,x(\theta)\} \in \cup_{N=t}\cap_{i,s}E_{i(s)}\{\theta,x(\theta)\}_n$, intertemporally and across systems and their variables by means of the IIE- learning processes and multiple relations signified by events E(.)

Inter-systemic interaction

$\{\theta,x(\theta)\} \in \cup_{N=t}\cap_{i,s}E_{i(s)}\{\theta,x(\theta)\}_N$
Inter-and intra-systems multiple variable i(s). 'Non-fixed point' properties of evolutionary spaces intra- and inter-systems are implied by the continuity of $\{\theta,x(\theta)\}$-values that make such learning spaces of the fuzzy-set type.

Mappings for wellbeing evaluation $\{W(\theta,x(\theta))\}\in\{\cup_{N=t}\cap_{i,s}W(E_{i(s)}(\theta,x(\theta))_N)$ by the property of toplogical preserving order of mappings on topological spaces of $\{\theta,x(\theta)$

$\{dW/d\theta > 0\in\{\cup_{N=t}\cap_{i,s}dW(E_{i(s)}(\theta,x(\theta))_N)/d\theta\}_t$
Subject to $(d/dt)(d\Delta E_s/d\theta^*) < 0$
In this paper the continuity of the topological mapping on monotonically continuous functional relations implies that order-preserving properties hold on the evolutionary learning case.

Figure 3. Geometry of evolutionary learning equilibriums.

APPENDIX 1
ON THE EXISTENCE OF EQUILIBRIUM IN NON-FIXED POINT' TYPE LEARNING SPACES

A geometrical explanation and proof is presented here to explain the total picture of the existence of knowledge-induced evolutionary equilibriums in a relevant form of the Non-Fixed Point Theorem. We deconstruct the following equation from the text of the paper that combines the entire nature of evolutionary equilibriums:
Dynamic equilibriums in evolutionary learning spaces are now defined by the following conditions:

$\exists\{\theta, x(\theta)\} \in \{E_{i(s)}(\theta, x(\theta))_n\}$, such that all the equations (1)-(3) in the paper hold true.

APPENDIX 2
ALGORITHMIC DISPLAY OF EVOLUTIONARY SYSTEMIC EQUILIBRIUMS

Simulate the wellbeing function given in the following form:

$W(\theta, x(\theta), AC(\theta), \varepsilon(\theta)) = \theta + x(\theta) + AC(\theta) + \varepsilon_N(\theta) > 0 \in \left\{\varepsilon_N(\theta) = \frac{1}{N}; N\right\}$ being numbered evolutionary learning systems}; $\frac{d\varepsilon(\theta)}{d\theta} < 0$ within the respective evolutionary regions in respect of an inestimable core convergent value $\varepsilon_N(\theta)^*$. The evolutionary values of $\varepsilon_N(\theta)$ do not compare between regions.

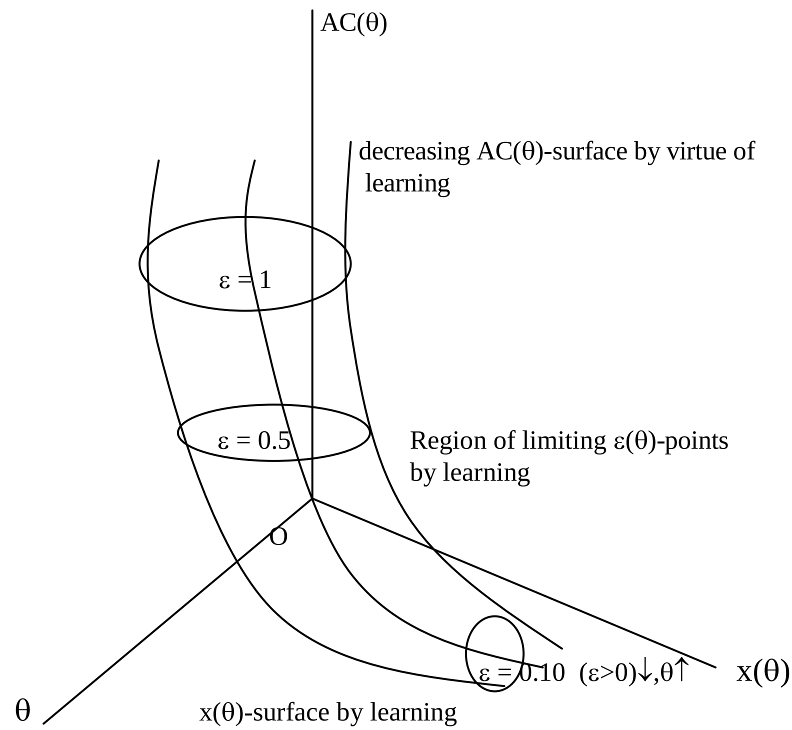


Figure 4. Evolutionary learning phenomenon relating to {θ,x(θ),AC(θ)}.

As an example, in the region with θ = 1, 2, 3, …10, leading respectively to $x(\theta)$ = 1,2,3,…10; $AC(\theta)$ = 1/1. 1/2, .., 1/10. The evolutionary system of wellbeing index is thus made up of regions bounded by the triplets, {(1,1,1, ε_1 (θ =1) = 1), (2,2,0.5, ε_2 (θ = 2) = 0.5), …, (10,10,0.10, ε_{10} (θ = 10) = 0.10)}.

Figure 4 shows the evolutionary learning phenomenon for the stated wellbeing function in reference to Figures 2 and 3.

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