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Model Calibration in Option Pricing

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ABSTRACT: We consider calibration problems for models of pricing derivatives which occur in mathematical finance. We discuss various approaches such as using stochastic differential equations or partial differential equations for the modeling process. We discuss the development in the past literature and give an outlook into modern approaches of modelling. Furthermore, we address important numerical issues in the valuation of options and likewise the calibration of these models. This leads to interesting problems in optimization, where, e.g., the use of adjoint equations or the choice of the parametrization for the model parameters play an important role.

KEYWORDS: Adjoints, Calibration, Jump models, Local volatility models, Mixed models, Partial differential equation (PDE), Stochastic differential equation (SDE), Stochastic volatility models.

معايرة النماذج في تسعير الخيارات

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ملخص: نفترض مسائل المعايرة لنماذج مشتقات التسعير التي تظهر في التمويل الرياضي. نناقش طرق متنوعة كاستخدام المعادلات التفاضلية العشوائية أو المعادلات التفاضلية الجزئية لعملية النمذجة. نعرض التطورات المنشورة مسبقا ونعطي آفاق حول الطرق الحديثه للمعايرة. إضافة إلى ذلك، نعالج مواضيع عددية هامة في تقييم الخيارات ومعايرة تلك النماذج. هذا يؤدي إلى مسائل مهمة في الأمثليات حيث، مثلآ، أن استخدام معادلات مساعدة أو اختيار معاملات تلعب دورا هاما في نموذج المعاملات.

1. Introduction

Financial derivatives, like options and futures, have gained considerable importance since the Chicago Board Options Exchange (CBOE), the first exchange to list standardized exchange-traded stock options, was founded in 1973. Starting with 911 contracts on 16 underlying stocks on the first trading day on April 26, 1973, the CBOE reported a total number of over 1.1 billion traded contracts in 2009, which corresponds to an average volume of more than 4.5 million contracts a day.[1] The rapid growth over the last 40 years of financial derivative markets, certainly owes its success to the publication of Black and Scholes (1973) and its extension by Merton (1973), since they laid the foundation of preference-free valuation of contingent claims. Particularly, they developed a simple, but powerful model that governs the price of European-style call and put options over time.

The main achievement, however, was not only the derivation of a valuation formula in closed form but also the idea of building a (hedge) portfolio by buying and selling the underlying asset and a risk-free bond in such a

[1]Source: http://www.cboe.com/aboutcboe/History.aspx & http://www.cboe.com/data/marketstats-2009.pdf

self-financing way that it perfectly matches the payoff (at maturity) of the option to be priced. Consequently, the amount of initial capital needed for building up the hedge portfolio coincides with the price of the considered European-style option. These publications form the cornerstone of today's financial industry.

But not only the total number of contracts, also the variety have grown in a remarkable way. Nowadays, in addition to standard European-style plain vanilla call and put options, exotic derivatives like digital or barrier options of American- or Bermudan-type, Asian-style options like lookbacks, or chooser options, cliquets or any reasonable combination, are frequently traded on financial derivative markets. However, it has been shown (in a variety of publications and text books) that due to its simplicity the classical Black-Scholes model cannot properly capture the real market dynamics. The Black-Scholes model is, unfortunately, not suitable to adequately price and hedge exotic options. In the literature, a multiple of models can be found subsequently relaxing the assumptions of the classical Black-Scholes model, for instance, by adding another degree of freedom to the process of the underlying asset.

In order to extract accurate market dynamic information to price and hedge exotic options, practitioners, e.g., traders and risk managers, need to adapt their models to the current market situation, i.e. the models have to be calibrated to a set of liquidly traded standard instruments like plain vanilla options.

The pricing of options as well as model calibration are interesting mathematical problems from various points of view. They pose challenges in several areas like mathematical modeling, stochastic processes, partial differential equations, optimization and numerical analysis. In Section 2, we briefly review the fundamentals of smile-consistent option pricing and its numerical pricing techniques like bi- and trinomial trees, Monte Carlo methods, and PDE pricing, for the case where no closed form solution is available. We will focus mostly on European-style call options and briefly discuss some pros and cons of the main classes of smile-consistent volatility models proposed in the literature. More precisely, we consider stochastic volatility models, local volatility models, jump models, as well as mixed volatility models and emphasize their relevance in practice. Section 3 gives an exhaustive survey of publications on the calibration of financial market models. Although several references on Monte Carlo calibration are given, we focus mostly on literature concerning the reconstruction of the local volatility function. We distinguish parametric and non-parametric approaches and briefly illustrate three categories of calibration procedures proposed in the literature. In doing so, we closely follow the distinction of Bouchouev and Isakov (1999), i.e. optimization-based algorithms, extra- and interpolation schemes, and iterative methods.

2. Option pricing

Starting with the Black-Scholes model, today's price of a *European-style call (put) option* with maturity T and strike K under some risk-neutral measure Q is defined as

$$\text{Call:}\quad C(S_0,0) = e^{-rT}\,{}^{Q}(\max(S_T - K, 0)), \tag{1}$$

$$\text{Put:}\quad P(S_0,0) = e^{-rT}\,{}^{Q}(\max(K - S_T, 0)), \tag{2}$$

where S_T denotes the asset price at maturity T given the asset price process $(S_t)_{0 \le t \le T}$ as a solution of the *Black-Scholes stochastic differential equation* (SDE)

$$dS_t = (r-d)S_t\,dt + \sigma S_t\,dW_t \tag{3}$$

with $S_0 \in (0,\infty)$ and $t \in [0,T]$. The (constant) instantaneous drift term consists of the risk-less interest rate r and the dividend yield d of the underlying.[2] Furthermore, σ denotes the (constant) instantaneous volatility function and $(W_t)_{0 \le t \le T}$ represents a Brownian motion (or Wiener process) defined on a probability space $(\Omega, \mathcal{F}, Q)$ with σ-algebra $\mathcal{F}$ over the set $\Omega \neq \emptyset$ and Q the unique risk-neutral measure (or martingale measure). The Brownian motion $(W_t)_{0 \le t \le T}$ is adapted to an adequate filtration $(\mathcal{F}_t)_{0 \le t \le T}$, where the filtration $(\mathcal{F}_t)_{0 \le t \le T}$

[2]For simplicity, we assume constant interest rates and dividend yields and further omit equity premiums.

satisfies some 'technical' conditions (see (Karatzas and Shreve, 2008) for details). As already mentioned, one of the key achievements of Black and Scholes (1973) was to provide an explicit valuation formula (known as the *Black-Scholes formula*[3]) for European-style call and put options, namely

$$C^{BS}(S_t, t; K, T; r, d, \sigma) = e^{-d(T-t)} S_t \phi(\delta_1) - e^{-r(T-t)} K \phi(\delta_2) \;, \tag{4}$$

$$P^{BS}(S_t, t; K, T; r, d, \sigma) = e^{-r(T-t)} K \phi(-\delta_2) - e^{-d(T-t)} S_t \phi(-\delta_1) \;, \tag{5}$$

where $$\delta_1 = \frac{\ln(S_t / K) + (r - d + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \qquad \delta_2 = \delta_1 - \sigma\sqrt{T-t}$$

with $\phi(x)$ the cumulative distribution function of the standard normal distribution, i.e.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \;.$$

Hence, the Black-Scholes option price at time $t \in [0,T]$ depends on the current value of the underlying S_t, i.e. the spot price, the time the option expires *T*, i.e. the maturity, the exercise price or strike price *K*, the interest rate *r*, dividend yield *d* and finally the (constant) volatility σ. A well-known model-free relationship between calls and puts on the same underlying asset, with equal strike and maturity, is the *put-call parity*

$$C^{BS}(S_t, t; \ldots) - P^{BS}(S_t, t; \ldots) = e^{-d(T-t)} S_t - e^{-r(T-t)} K \;.$$

It is easy to show that given a fixed Black-Scholes price $\bar{C}$ satisfying reasonable non-arbitrage conditions, i.e. $\max(e^{-d(T-t)} S_t - K e^{-r(T-t)}, 0) \le \bar{C} \le e^{-d(T-t)} S_t$, the mapping

$$\sigma \to C(\sigma) - \bar{C}$$

has a unique root σ^{impl}, called *implied volatility*. Conversely, in the classical Black-Scholes model the option price $C^{BS}(S_t, t; K, T; r, d, \sigma^{impl})$ depends uniquely on its implied volatility σ^{impl}, where σ^{impl} is assumed to be constant in time *t*, stock price S_t, strike *K* and maturity *T*. This assumption, however, cannot be observed on the market. If one plots the observed market implied volatility against the strike *K*, the resulting graph will usually be downward sloping in equity markets, while it is typically valley-shaped in currency markets or for equity index options. This behavior is referred to as *'volatility skew'* or *'volatility smile'*, respectively. Furthermore, it can be observed that the volatility skew or smile usually flattens for long term maturities. The change of implied volatilities with respect to different maturities is called *'term structure'* of the implied volatility surface. Finally, the implied volatility surface also changes dynamically over time. A more detailed introduction to this topic is given, e.g., in (Hull, 2011).

Although, the classical Black-Scholes model lacks on realism, implied volatility serves as a standardized (or normalized) value (usually quoted in %) of market volatility. In sticky-moneyness markets[4], the implied volatility provides more stability than the Black-Scholes option price. Practitioners use implied volatility as a language, rather than as a model.

A lot of research has been done over the last 40 years, trying to explain this strike deviation from the Black-Scholes constant volatility assumption. Many factors have been investigated as being possibly responsible for the smile and term structure of the implied volatility surface. They range from the existence of transaction costs or liquidity constraints, to stochastic volatility and jump processes for the underlying asset price process. In the following we focus on the latter ones.

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[3]In fact, Black and Scholes (1973) derived their valuation formula by solving the *Black-Scholes partial differential equation*, which will be introduced later on.

[4]Moneyness is defined as the quotient between stock price S_t and strike price *K*.

2.1 Smile-consistent pricing models

The idea behind the development of new pricing models, so called *'smile-consistent pricing models'*, is to directly extract information about the asset price and volatility dynamics from frequently traded standardized plain vanilla options in order to price and hedge exotic options. This is done by assuming the coefficients of (3) to be some deterministic function of the spot price S_t and the time *t*, by adding new sources of randomness or by adding all of it. Since the sources of randomness are usually added to the volatility [cf. Gatheral (2006)] the generalized framework (or extension) of the Black-Scholes model is given by replacing (3) by

$$dS_t = a(S_t, t) S_t \, dt + b(S_t, t) S_t \, dW_t \tag{6}$$

with $S_0 \in (0, \infty)$. The asset price $(S_t)_{0 \le t \le T}$ is, therefore, modeled by a $(\mathcal{F}_t)_{0 \le t \le T}$ adapted stochastic process, driven by the SDE (6), where $a(S_t, t)$ and $b(S_t, t)$ are the instantaneous drift and the volatility, respectively. Fengler (2005) assumed that the instantaneous volatility $b(S_t, t, \xi_t)$ follows some $(\mathcal{F}_t)_{0 \le t \le T}$ adapted arbitrarily depending stochastic process, where the 'ξ_t-dependence' simply emphasizes that $b(S_t, t, \xi_t)$ may also depend either on the history of S_t, i.e. $\xi_t = S_{t_N}, \cdots, S_{t_1}$, for $0 \le t_n < t$ for all *n*, or some other sources of randomness. Furthermore, the absence of arbitrage, i.e. the existence of some risk-neutral measure under which a discounted asset price $(S_t)_{0 \le t \le T}$ is a martingale, is assumed. However, the martingale measure needs not to be unique (see (Björk, 2004, p. 150)). To the knowledge of the authors, at least two main lines can be identified. *'Stochastic volatility models'* impose a new source of randomness to the volatility, while *'local volatility models'* treat the volatility as a deterministic function of S_t and *t*. Adding jumps as a new source of randomness leads to *'jump diffusion models'*. Most recently, *'stochastic local volatility models'* have become very attractive, since they allow a perfect fit while preserving the advantages of stochastic volatility models.

2.1.1 Stochastic volatility models

The most prominent stochastic volatility (SV) model was introduced by Heston (1993),

$$\begin{aligned} dS_t &= (r - d) S_t \, dt + \sqrt{v_t} S_t \, dW_t^s, \\ dv_t &= \kappa(\bar{v} - v_t) dt + \eta \sqrt{v_t} \, dW_t^v \end{aligned} \tag{7}$$

with $S_0, v_0 \in (0, \infty)$ and $\left\langle dW_t^s, dW_t^v \right\rangle = \rho$,

where $r - d$ denotes the (deterministic) instantaneous drift of stock price returns, η the volatility of volatility, κ the speed of reversion of v_t to its long-term mean $\bar{v}$ and ρ the correlation between W_t^s and W_t^v, the Brownian motions driving the stock price process and variance process, respectively. The latter one is a special case of the square root process proposed by Cox *et al*. (1985) to model interest rates, known as the CIR model. Other stochastic volatility models have been proposed in the literature, e.g., (Hagan *et al*., 2002, SABR; Bollerslev, 1986, GARCH; Stein and Stein, 1991), just to name a few of them. However, it is proven, for instance, in (Duffie *et al*., 2000) that the Heston model has a quasi-closed form solution for European-style options. Although other, possibly more realistic, stochastic volatility models are available, the existence of a quasi-closed form solution contributed substantially to the outstanding success of Heston's model in practice. This is due to the fact that computational efficiency becomes essential when calibrating the model to observed market data. Furthermore, according to Gatheral (2006), all stochastic volatility models generate roughly the same shape of implied volatilities and thus the same implications for the valuation of non-vanilla derivatives. On the other hand, he shows that, although the Heston model fits observed implied volatility for longer expirations, this is, unfortunately, not true for shorter-dated options. This motivates the use of *'jump diffusion models'*, since, loosely speaking, introducing jumps has very little impact for long term maturity options, while jumps are strongly noticeable in terms of implied volatility for short-expiration options.

2.1.2 Adding jumps

Jump diffusion models were first considered by Merton (1973) and later by Kou (2002) as an extension of one-dimensional processes. Duffie *et al*. (2000) proved that *'affine jump diffusion'* (AJD) processes, which consist, roughly speaking, of a jump diffusion process for which the drift vector, the 'instantaneous' covariance matrix, and the jump intensities all have affine dependence on the state vector, are in general analytically tractable. In case of a two-dimensional process we have

$$\begin{aligned} dS_t &= (r-d-k\lambda)S_t\,dt + \sqrt{v_t}\,S_t\,dW_t^s + d\left(\sum_{n=1}^{N_t} S_{\tau_n-}\left[e^{Z_n^s}-1\right]\right), \\ dv_t &= \kappa(\bar{v}-v_t)dt + \eta\sqrt{v_t}\,dW_t^v + d\left(\sum_{n=1}^{N_t} Z_n^v\right) \end{aligned} \tag{8}$$

with $S_0, v_0 \in (0,\infty)$ and $\left\langle dW_t^s, dW_t^v \right\rangle = \rho$,

where r, d, η, κ, $\bar{v}$, and ρ are as in (7). Further, N_t denotes a Poisson process with jump intensity $\lambda > 0$ and drift compensation k. The jump sizes in stock price and volatility, i.e. Z_n^s and Z_n^v, respectively, for $n = 1, \cdots, N_t$, are assumed to be i.i.d. (independent and identically distributed). Finally, $S_{\tau_n-} := \lim_{t \nearrow \tau_n} S_t$ denotes the stock price at τ_n right before the jump occurs.

Thus, (8) denotes a stochastic volatility model with simultaneous jumps in stock price and volatility (SVJJ). Note that Heston's stochastic volatility model is a jump-free special case, i.e. $Z_n^s = Z_n^v = 0$ for all *n*, of an AJD process and as such has at least a quasi-closed form solution as mentioned before.[5] The Merton and Heston approaches were combined by Bates (1996), who proposed a model with stochastic volatility and jumps (SVJ). Bates' model is also incorporated in (8) as a special case, where $Z_n^v = 0$ for all *n*. Gatheral (2006) shows that SVJ models perform empirically as well as SVJJ, but they have less parameters. Therefore, SVJ models like Heston's model are frequently used in practice. An extensive discussion about jump diffusion models can be found in (Cont and Tankov, 2004).

We now turn our focus onto *'local volatility (LV) models'*. They became quite popular in the past due to their simplicity, however, they have also gained a lot of criticism in financial literature (cf., e.g., (Ayache *et al*., 2004; Hagan *et al*., 2002)).

2.1.3 Local volatility models

Following Fengler (2005), '*local variance'* may be defined as the risk-neutral expectation of the instantaneous variance conditional on $S_T = K$ and time filtration $\mathcal{F}_t$, i.e.

$$\hat{\sigma}_{K,T}^2(S_t,t) := \mathrm{E}^Q\left(b^2(S_T,T,\xi_T)|S_T = K, \mathcal{F}_t\right),$$

where $b(S_t,t,\xi_t)$ is as before. Then, *'local volatility'* (also called *'forward volatility'*) is given as the square root of local variance. The main advantage of this definition of local volatilities is that it naturally implies the purely deterministic case, but also offers some insights into the concept of stochastic volatility. Within this framework of local volatilities, for some market level $K = S_t$ at $T = t$, the instantaneous volatility is given by

$$\sigma(S_t,t) = \hat{\sigma}_{S_t,t}(S_t,t),$$

such that (with $a(S_t,t) := r-d$, for simplicity)

$$dS_t = (r-d)S_t\,dt + \hat{\sigma}_{S_t,t}(S_t,t)S_t\,dW_t \tag{9}$$

[5] In fact, the Heston's formula is given as a linear combination of two integrals of real-valued functions.

defines the stock price process, which generalizes the classical Black-Scholes theory as desired. The intrinsic stochasticity is integrated out and we are left with a one-factor diffusion process. However, if by assumption~~s~~ the instantaneous volatility is deterministic in spot and time, i.e. $b(S_t, t, \xi_t) = \sigma(S_t, t)$, both concepts of instantaneous and local variance coincide, since

$$\begin{aligned}\hat{\sigma}^2_{K,T}(S_t, t) &:= \mathrm{E}^Q(b^2(S_T, T, \xi_T)|S_T = K, \mathcal{F}_t) \\ &= \mathrm{E}^Q(\sigma^2(S_T, T)|S_T = K, \mathcal{F}_t) = \sigma^2(K, T)\ .\end{aligned}$$

The local volatility assumption is the easiest way of relaxing the constant volatility case and it introduces much more flexibility. In contrast to stochastic volatility models, the concept of local volatility preserves the assumption of market completeness.[6] Originally, Dupire (1994) and Derman and Kani (1994a)[7] have shown that given the distribution of the final stock price S_T for each time T conditional on some starting price S_0, there exists a unique risk-neutral diffusion process (9) consistent with these distributions. The reason is that there exists a *'dual'* or *'adjoint'* PDE to the classical Black-Scholes PDE (cf. Section 2). The remarkable observation that local volatility can be seen as the market expectation of future volatility, known as *'Markovian projection'*, was independently derived by Dupire (1996) and by Derman and Kani (1998).

Different assumptions on the special shape of the local volatility function have been made in the literature. They are either motivated by model calibration in order to reduce the number of unknowns (see Section 3.3.2), or by empirical observations (see, e.g., (Dumas *et al*., 1998; Coleman *et al*., 2001)) in order to properly capture the dynamics of the underlying asset. A prominent example is the *'constant elasticity of variance model'* (CEV) introduced by Cox and Ross (1976), where $\sigma(S_t, t) = \sigma S_t^{\gamma-1}$ with $\sigma, \gamma > 0$. The CEV model attempts to heuristically capture the stochastic volatility, where γ controls the relationship between volatility and price. When $\gamma < 1$, commonly observed in equity markets, the volatility of the underlying increases as its price falls. Conversely, in commodity markets, the volatility of the underlying tends to increase as its price increases. Note that for $\gamma = 1$ we obtain the Black and Scholes case.

Ingersoll (1997) and Rady (1997) introduced the class of bounded quadratic diffusion models, i.e. $\sigma(S_t, t)$ which is considered to be a bounded quadratic function in asset price and/or time. Zühlsdorff (2001) has proven the existence and uniqueness of the solution of the underlying SDE and provided explicit formulas for call options assuming that the deterministic local volatility function can be split in a strictly positive and bounded function γ and a quadratic polynomial p such that $\sigma(S_t, t) = p(S_t)\gamma(t)$. Option pricing in the quadratic volatility model is a rather delicate issue, since it touches the limits of no-arbitrage theory. Andersen (2011) clarified some confusion in literature and further extended the range of existing pricing formulas. Coleman *et al*. (2001) published empirical evidence that a spline representation can provide a more accurate representation in terms of hedging compared to the quadratic model considered by Dumas *et al*. (1998).

Although the deterministic local volatility function may look very complicated, considering local volatilities can be a questionable model simplification. Ayache *et al*. (2004) and Hagan *et al*. (2002) doubt that a one-factor diffusion model delivers an adequate description of the asset price behavior. Hagan *et al*. (2002) illustrated that the model delta of deterministic local volatilities is wrong or at best very misleading. This, however, is a crucial issue in terms of the dynamic hedging performance of the model. Furthermore, another undesirable feature of the local volatility model is that it predicts flat future smiles, such that forward-start options or cliquets are likely to be mispriced. Beside these pricing and hedging problems, Ayache *et al*. (2004) criticized that local volatility models reveal no reasonable explanation for the existing smile phenomenon.

Despite all criticism, local volatility models are widely used in practice. Common problems arising from using complex models like *'jump diffusion models'* or even *'mixed volatility models'* are the additional

[6]Note that volatility is not a tradeable asset, which implies that the completeness of the market, i.e. the ability to hedge options with the underlying asset only, is lost.
[7]While Dupire (1994) developed a continuous time theory, Derman and Kani (1994a) used a discrete binomial tree approach.

computational effort, the high implementation costs, the loss of intuition, and a potential decrease in calibration stability. Hence, *'practitioners may, and in fact often do, favor a simple and intuitive model'*, see (Coleman *et al*., 2011). Furthermore, it is most likely that Dupire (1994) and Derman and Kani (1994a) did not introduce the local volatilities as a model of its own, but instead they intended to propose an intuitive way to price exotic derivatives under certain market circumstances.

2.1.4 Mixed or hybrid volatility models

As jump processes have been added to stochastic volatility models to provide a better fit of model implied volatilities to market implied volatilities (especially for short term maturities), the local volatility framework has been applied to stochastic volatility models. So-called *'stochastic local volatility (SLV) models'* were proposed by Blacher (2001) and Lipton (2002) and were studied further, e.g., in (Ren *et al*., 2007; Piterbarg, 2007; Alexander and Nogueira, 2008; Henry-Labordère, 2009). As an example, the governing SDEs for a *'Heston-type stochastic local volatility model'* are:

$$\begin{aligned} dS_t &= (r-d)S_t\,dt + \sigma_{LV}(S_t,t)\sqrt{v_t}\,S_t\,dW_t^s \\ dv_t &= \kappa(\bar{v}-v_t)dt + \eta\sqrt{v_t}\,dW_t^v \end{aligned} \tag{10}$$

with $S_0, v_0 \in (0,\infty)$ and $\langle dW_t^s, dW_t^v \rangle = \rho$,

where again r, d, η, κ, $\bar{v}$, and ρ are as in (7) and σ_{LV} denotes the local volatility function. Then, in the framework of local variance, the instantaneous 'hybrid' variance takes the form:

$$\hat{\sigma}^2_{K,T}(S_t,t) = \sigma^2_{LV}(K,T)\mathrm{E}^Q(v_T \mid S_T = K, \mathcal{F}_t)\,. \tag{11}$$

Because of this particular form of (11), it is not possible to separate the influence of the stochastic component from the local component in an intuitive manner. Thus, Tavella *et al*. (2005) prefer to define the instantaneous hybrid volatility as a weighted sum of a stochastic component and a local component. It is worth mentioning that Lipton (2002) and Lipton and McGhee (2002) further extended (10) by adding jumps to the stock price process S_t. Among others, this extension of (10), called the *'universal model'*, was strongly criticized by Ayache *et al*. (2004). It is argued that, roughly speaking, there is no chance to reveal the true market smile dynamics, since the freedom of the local volatility function can nearly compensate every dynamics introduced by the stochastic or jump component. In practice, this problem is usually addressed by separately calibrating the model parameters to extract plausible dynamics from the market.

2.2 Numerical evaluation of smile-consistent pricing models

Fast model evaluation is a crucial issue in practice. In order to be competitive with other market participants, very complex derivatives need to be priced nearly on-the-fly. Additionally, a fast and stable pricing scheme is essential when calibrating a financial market model to a large number of market data. Therefore, it is not surprising that models, which provide a closed or quasi-closed form solution, have become popular in practice. A survey of most of the existing market models with closed (or quasi-closed) form solutions has been given , e.g., in (Kolb and Overdahl, 2010, Chap. 27; Hull, 2011; Andersen, 2011), - especially for unbounded quadratic local volatility models. In the early years, bi- or trinomial trees have been a typical approach to price path-independent and path-dependent options in consistence with the prevailing volatility smile. This valuation method, which can be seen as a discrete version of Black-Scholes pricing PDE, was pioneered by Cox *et al*. (1979) (CRR).[8]

A very natural way to price complex derivatives are Monte Carlo or quasi-Monte Carlo methods. They are based on the continuous time models, i.e. the fundamental pricing formula (1) or (2) and the considered market model:

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[8]More precisely, it can be easily shown that, for instance, the trinomial method is an example of an explicit finite difference scheme of Black-Scholes pricing PDE and therefore it inherits certain stability properties of finite difference methods, cf. (Duffy, 2006, Chap. 13)

$$\begin{aligned} C(S_0,0) &= e^{-rT}\,\mathrm{E}^Q\left(\varphi(S_T,\zeta_T)\right) \\ \text{s.t.}\quad dS_t &= a(S_t,t)S_t\,dt + b(S_t,t,\xi_t)S_t\,dW_t\,,\quad S_0\in(0,\infty)\,, \end{aligned} \tag{12}$$

where $\varphi(S_T,\zeta_T)$ denotes the payoff function, depending on the asset price S_T at maturity T and possibly on some history $\zeta_T = S_{t_N},\cdots,S_{t_1}$ $(0\le t_n < T$ for all n). As discussed, the general market model in (12) may be replaced by a specific one like (7), (8), (9), or (10). Thus, in case of a European-style call option under local volatility we obtain

$$\begin{aligned} C(S_0,0) &= e^{-rT}\,\mathrm{E}^Q\left(\max(S_T-K,0)\right) \\ \text{s.t.}\quad dS_t &= (r-d)S_t\,dt + \sigma(S_t,t)S_t\,dW_t\,,\quad S_0\in(0,\infty)\,. \end{aligned} \tag{13}$$

In order to generate Monte Carlo samples, the stochastic differential equation in (14) is usually discretized by a Euler-Maruyama or Milstein scheme (see (Kloeden and Platen, 1999)) which is then used to approximate the expected value functional. Following the well-known law of large numbers, the pricing problem (14) can be formulated as

$$\begin{aligned} C(S_0,0) &\approx e^{-rT}\,\frac{1}{M}\sum_{m=1}^{M}\max(s_N^m-K,0) \\ \text{s.t.}\quad s_{n+1}^m &= s_n^m + (r-d)s_n^m\,\Delta t_n + \sigma(s_n^m,t_n)s_n^m\,\Delta W_n^m\,, \\ s_0^m &= S_0,\quad n=0,\cdots,N-1,\quad m=1,\cdots,M\,, \end{aligned} \tag{14}$$

where M, sufficiently large, denotes the number of random samples. Further, s_n^m is the m-th realization of the solution of (9) at time t_n given a time discretization $0=t_0<t_1<\cdots<T$ with step size $\Delta t_n := t_{n+1}-t_n$ for $n=0,\cdots,N-1$. Accordingly, $\Delta W_n := (W_{n+1}-W_n)$ are the discrete increments of the Brownian motion W_t at time t_n.

Due to their flexibility to changes in either the payoff function or the considered market model, (quasi)-Monte Carlo methods are widely used in practice. They further allow the computation of path-dependent derivatives with relatively small extra costs. Moreover, Monte Carlo methods are very easy to parallelize, such that the use of modern graphics processors (GPUs) allows a tremendous speed-up in computation time. Consequently, high dimensional problems can be solved in a reasonable time. An exhaustive discussion of (quasi)-Monte Carlo methods can be found in (Glasserman, 2003).

While (quasi)-Monte Carlo methods are quite meaningful in high dimensions, they can be very slow for low dimensional problems. Representation theorems like Feynman-Kac's theorem (cf. (Karatzas and Shreve, 2008)) show that, for instance, in the case of European-style call options, the price process of a plain vanilla call option follows some parabolic partial differential equation, i.e.

$$\begin{aligned} C(S,t) &= -\frac{1}{2}S^2\sigma(S,t)^2C(S,t) - (r-d)SC(S,t) + rC(S,t)\,, \\ &\qquad (S,t)\in(0,\infty)\times[0,T)\,, \\ C(S,T) &= \max(S-K,0),\quad S\in(0,\infty)\,. \end{aligned} \tag{15}$$

To put it differently, under reasonable assumptions of the existence of unique solutions of the local volatility model, i.e. SDE (9) and Black-Scholes pricing PDE (15), the unique solution $C(S_t,t)$ of (15) admits for all $S_t\in(0,\infty)$ and $t\in[0,T]$ the stochastic representation

$$\begin{aligned} C(S_t,t) &= e^{-r(T-t)}\,\mathrm{E}^Q\left(\max(S_T-K,0)\right) \\ \text{s.t.}\quad dS_\tau &= (r-d)S_\tau\,d\tau + \sigma(S_\tau,\tau)S_\tau\,dW_\tau\,,\quad S_t\in(0,\infty) \end{aligned} \tag{16}$$

with $\tau\in(t,T]$. Similar results can be obtained for, among others, barrier-, digital-, or plain vanilla call and put options with local or stochastic volatility. While basket options and models with stochastic volatility and/or stochastic interest rate lead to multidimensional PDEs (see (Wilmott, 2006)), an integral term is added in the PDE when considering jump diffusion models (see (Cont and Tankov, 2004)). Thus, the latter one requires the numerical solution of a partial integro-differential equation (PIDE). Pricing American-style options yields the challenge of solving free-boundary value problems (see (Wilmott, 2006)). Asian-style options can be modeled using one- or two-dimensional PDEs, depending on special payoff characteristics (see (Zvan *et al*., 1998)). In either case, numerical methods like finite difference or finite element methods need to be applied, when no analytic solution is available. Among multiple standard textbooks on numerical methods for PDEs, an exhaustive discussion on robust, accurate and efficient finite difference methods particularly for pricing various derivative products can be found in (Duffy, 2006;Tavella and Randall, 2000). Topper (2005) and Achdou and Pironneau (2005) focus on finite element methods used in quantitative finance. A current survey of efficient numerical methods for solving those types of PIDEs is given in (Feng and Linetsky, 2008; Sachs and Strauss, 2008).

Beside the fact that under risk-neutrality there is a unique diffusion process consistent with the prevailing market smiles, Dupire (1994) and Derman and Kani (1994a) discovered that European-style option prices in the local volatility model satisfy a certain forward PDE, in which the independent variables are the options' strike K and maturity T, i.e.

$$\begin{aligned} &D(K,T) = \frac{1}{2}K^2\sigma(K,T)^2 D(K,T) - (r-d)KD(K,T) - d\,D(K,T)\,, \\ &(K,T) \in (0,\infty)\times(0,T_{\max}]\,, \\ &D(K,0) = \max(S_0 - K,0), \quad K \in (0,\infty)\,. \end{aligned} \tag{17}$$

This forward evolution equation, called *'Dupire's equation'*, is of twofold importance: (i) Dupire's equation in a local volatility framework is used to explicitly determine the underlyings' instantaneous volatility function, see Section 3. (ii) Once volatility function is known, the forward PDE can be solved numerically to efficiently price a collection of European-style options of different strikes and maturities all written on the same underlying asset.

Due to its significance in option pricing and model calibration, several extensions, also called *'forward equations'*, have been proposed in financial literature. Andersen and Andreasen (1999) derived and Andreasen and Carr (2002) further extended, forward equations for European-style options in jump diffusion models. It is straightforward to develop the relevant forward equation for barrier or digital options (see, e.g., (Pironneau, 2006; Pironneau, 2007; Carr and Hirsa, 2007)). Buraschi and Dumas (2001) give a forward representation of compound option prices for general diffusion processes with deterministic volatility. Forward equations for American-style put options with jump diffusion processes are derived in (Carr and Hirsa, 2003; Amster *et al*., 2009) and are considered Dupire-like equations for multi-asset options, while Bentata and Cont (2010) further generalized Dupire's forward equation to a large class of non-Markovian models with jumps.

Basically two lines of techniques to derive Dupire-like forward equations can be found in the literature. Either a generalization of Itô's formula, called *'Tanaka-Meyer formula'* (see (Karatzas and Shreve, 1998, p. 220)), is applied to the underlying SDE, or classical adjoint calculus is used to formulate the *'dual'* problem of Black-Scholes pricing PDE. The first approach is explained, for instance, in (Bentata and Cont, 2010), while adjoint calculus (see, e.g., (Friedman, 1964)) is applied in (Pironneau, 2006; Pironneau, 2007; Pironneau, 2009). A comparison of both approaches for the original Dupire's equation can be found in (Fengler, 2005).

3. Model calibration

One of the key issues in quantitative finance is model calibration. Practitioners, like traders or risk managers, need to extract accurate market dynamic information, in order to correctly price and hedge derivatives. Usually, this is done by calibrating the relevant financial market model to a set of frequently traded standard instruments. In this section, we shortly review the common techniques used in practice.

Due to its considerable importance, we start with restricting our attention to the calibration of local volatility models. In the literature, mainly three groups of approaches are proposed: extra- and interpolation schemes, iterative procedures on analytic approximations, and optimization-based methods.[9] In the last part, however, we extend our view to the calibration of other models as well, since optimization-based methods naturally allow more flexibility in terms of changing the considered model.

3.1 Extra- and interpolation techniques

The concept of *'implied bi- and trinomial trees'* is aligned with the option pricing scheme introduced by Cox *et al*. (1979). Instead of setting up a smile-consistent pricing tree in advance using, e.g., parameters inferred from a separate calibration routine, implied trees are directly recovered from observed market data approximating all necessary risk-neutral transition probabilities. Binomial trees, proposed by Derman and Kani (1994b) and Barle and Cakici (1998), however, suffer from a number of fundamental problems. According to Boyle and Lau (1994), binomial trees may encounter unpredictable convergence behavior when pricing options with discontinuous payoffs (like barriers or digitals). Secondly, and more importantly, negative transition probabilities may occur, from which arbitrage opportunities ensue. In contrast to Derman and Kani (1994b) and Barle and Cakici (1998), who construct the tree using a forward recursion formula, Rubinstein (1994) and Jackwerth (1997) propose to inductively build up the tree beginning from a risk-neutral distribution at the terminal node. This, by construction, prevents the probabilities becoming negative. Derman and Kani (1994b) suggest the use of trinomial trees, which are somehow equivalent to a simple explicit finite difference approximation of Dupire's equation (17). Trinomial trees provide a more flexible approximation to the state space than a binomial tree, but suffer from many of the same problems as the binomial tree and are prone to instability. Andersen and Brotherton-Ratcliffe (1997/1998), therefore, use the well-known (semi-implicit) Crank-Nicolson approximation, since it exhibits much better stability and convergence properties.

In any tree or tree-related approach, it is assumed that plain vanilla options are available for every strike and time to maturity. Therefore, option prices need to be inter- and extrapolated into regions where no market data are observable. This becomes even more relevant when considering the time continuous theory.

Having built the bridge to Dupire's equation (17) earlier, practitioners sometimes prefer a slightly different view on (17), i.e.

$$\sigma(K,T)^2 = 2\frac{D(K,T)+(r-d)KD(K,T)+d\,D(K,T)}{K^2 D(K,T)}\,. \tag{18}$$

In this approach, calibration is meant to find a continuous and sufficiently smooth option price function $D(K,T)$ with respect to strike K and maturity T, such that the local volatility function can be recovered from (18), known as *'Dupire's formula'*. Therefore, again some inter- and extrapolation techniques are required, in order to obtain a continuous and smooth price function for market data information. Aside from the smoothness requirements, the challenge is to guarantee that no standard arbitrage bounds are violated and that the local variance remains positive and finite.

Due to the fact that, under no-arbitrage conditions, there is a unique implied volatility given an option price and vice versa (see Section 2), the necessary inter- and extrapolation is usually done on the implied volatility side. Kahalé's interpolation procedure (see (Kahalé, 2004)), for instance, is based on piecewise convex polynomials, which mimics the Black-Scholes formula. If the input data are free of arbitrage, so will be the resulting implied volatility surface. Fengler (2009) used natural cubic splines in space and finite differences in time to parametrize the local volatility function. The idea is to solve a sequence of small quadratic programs (QP) arising from a least-squares formulation (LSQ) of the spline representation under additional no-arbitrage constraints. Calender arbitrage is avoided by imposing transfer conditions through the iterates of the sequence of QPs. In contrast to Kahalé (2004), the main advantage of Fengler (2009) is that the input data need not to be free of arbitrage. Benko *et al*. (2007) suggest estimating the implied volatility surface with local quadratic polynomials. Arbitrage is ruled out by forcing the state-price density to be non-negative. Hanke and Rösler (2005) solve a normal equation to get an equivalent minimal norm solution of a LSQ problem using the

[9] See also (Bouchouev and Isakov, 1999).

discretized Dupire's formula and cubic splines in space and finite differences in time to parametrize the local volatility function. An extensive review about useful smoothing techniques is given in (Fengler, 2005, Chap. 4).

Calibration using Dupire's formula (18) requires an interpolation method for either the implied volatility surface or the option price function. However, practitioners have stated that the resulting local volatility surface is very unstable and that the option prices are very sensitive to the interpolation method (see, e.g., (Lipton, 2001)). Furthermore, a difficulty is the extra- and interpolation into areas where no market data are observed. For instance, it is not clear how to extrapolate prices for options that mature before the closest expiration date.

3.2 Analytic approximations (and iterative methods)

Bouchouev and Isakov (1999) apply the classical parametrix technique for PDEs to derive an analytic approximation of the option premium as a sum of its Black-Scholes price and an integral correction for the non-constant volatility, which explicitly shows the nonlinear relationship between the option price and volatility. The resulting integral equation is then discretized and iteratively solved at the points where observed option prices are available. A second iterative algorithm is applied directly to the fundamental solution of the underlying PDE. Both algorithms proposed are straightforward to implement. However, they should only be used for short term maturities, since the algorithm is applied for a single time period and then repeated for all consecutive maturities. Therefore, it might not be able to capture the term structure of the local volatility surface. Analytic approximation formulae for one-dimensional local volatility models using the parametrix methods are also derived in (Corielli *et al*., 2010), but not considered in the context of calibration.

3.3 Optimization-based calibration

While usually very fast, the methods described above are only applicable to a quite small class of financial market models. Optimization-based calibration methods offer more flexibility and are applicable to almost any financial market model. Trying to minimize the distance between *model data* and observed *market data*, for instance, in a least-squares formulation (LSQ), i.e.

$$\min_{a\in\mathcal{H}} \frac{1}{2}\sum_{m=1}^{M} (C_m^{mod}(a;S_0,0) - C_m^{obs})^2, \tag{19}$$

the model prices can either be given in closed form or as a solution of the underlying SDE or PDE model. Nonlinear constrained and unconstrained optimization problems arise in many sciences, and numerous methods exploiting the special structure have been proposed to efficiently solve them. An introduction to state-of-the-art numerical methods for constrained and unconstrained optimization problems can be found in (Nocedal and Wright, 1999).

As an example, to calibrate the time-dependent Heston call option model, Gerlich *et al*. (2010) designed a special sequential quadratic programming (SQP) optimization algorithm for the constrained least-squares problem (19), where $a \in \mathcal{R}^n$ denotes the vector of Heston parameters, $C_m^{mod}(a;S_0,0)$ and C_m^{obs}, for $m = 1,\cdots,M$, and the model, respectively, market data with maturity T and strike K given an initial stock price S_0 of the option's underlying at $t_0 = 0$. The feasible set $\mathcal{H}$ arises from additional constraints on the Heston parameters a, for example, by assuming that the variance of the underlying stochastic process remains strictly positive. The model prices $C_m^{mod}(a;S_0,0)$, for $m = 1,\cdots,M$, are evaluated using the closed form solution.

As mentioned before, closed form solutions are only available for a small class of models. In the following, we consider the least-squares calibration of financial market models in a more general setting.

3.3.1 SDE constrained optimization (Monte Carlo calibration)

Since financial market models are often characterized by the SDE of their underlying asset, calibration problem (19) may be written (for example, for a local volatility model) as

$$\min_{\sigma\in\mathcal{H}} f(\sigma) := \frac{1}{2}\sum_{m=1}^{M}(C_m^{mod}(\sigma;S_0,0)-C_m^{obs})^2$$
$$\text{where}\quad C_m^{mod}(\sigma;S_0,0)=e^{-rT}\,\mathrm{E}^Q\left(\max(S_{T_m}-K_m,0)\right), \tag{20}$$
$$\text{s.t.}\quad dS_\tau=(r-d)S_\tau d\tau+\sigma(S_\tau,\tau)S_\tau dW_\tau,\quad S_0\in(0,\infty),$$

where $\mathcal{H}$ is an appropriate set of volatility functions. SDE constrained optimization problems have not been much considered in financial market literature, since Monte Carlo methods are known to be very slow. Giese *et al*. (2007), for instance, use the Euler-Maruyama scheme to simulate the underlying stock price process in the Heston framework and combine it with a multi-layer method related to multigrid methods from the solution of PDEs as proposed, e.g, in (Kelley and Sachs, 1994). Further acceleration is achieved by parallelization of the price evaluations.

Recently, adjoint techniques have proven to be quite successful to accelerate Monte Carlo pricing. Adjoint techniques have their origin in the field of optimal control (see (Giles and Pierce, 2000)) and were recently introduced into finance in the context of automatic differentiation to compute derivatives (greeks) (see (Giles and Glasserman, 2006)). Kaebe *et al*. (2009a) applied adjoint calculus combined with a multi-layer approach to gain a tremendous speed-up in the calibration of a very general setting of systems of SDEs. Furthermore, Kaebe *et al*. (2009b, unpublished data) extended their consideration of Monte Carlo calibration of SDE models by adding jump components and proposed the use of a semi-analytical adjoint framework, which is based on a decomposition of the sensitivity computation into a sufficiently smoothed diffusion and jump part. Groß and Sachs (2011, unpublished data) derived an adjoint approach for the Milstein scheme (a second order discretization scheme) in connection with predictor-corrector methods. The feasibility and efficiency of Monte Carlo calibration for financial market models is discussed in (Käbe, 2010).

Monte Carlo calibration using adjoints provides at least two main considerable advantages. Beside its flexibility, it allows a strong parallelization and thus the use of GPUs (see Section 2) to gain remarkable speed ups. Secondly, second order information can easily be obtained due to pathwise use of adjoints (cf. (Kaebe *et al*., 2009a)). Hence, the number of necessary optimization iterations can be reduced tremendously.

3.3.2 PDE constrained optimization

We again consider the nonlinear LSQ problem (19) as an example for the calibration problem using the PDE framework (15), i.e.

$$\min_{\sigma\in\mathcal{H}} f(\sigma) := \frac{1}{2}\sum_{m=1}^{M}\left(C_m^{mod}(S_0,0)-C_m^{obs}\right)^2 \tag{21}$$
$$\text{s.t.}\quad C_m=-\frac{1}{2}S^2\sigma(S,t)^2C_m-(r-d)SC_m+rC_m,\quad (S,t)\in\Omega,$$
$$C_m(S,T_m)=\max(S-K_m,0),\quad S\in(0,\infty),\quad\text{for}\quad m=1,\cdots,M.$$

with $\Omega=(0,\infty)\times[0,T_{\max})$. A drawback of (21) is the fact that *M* PDEs need to be solved in order to obtain one function evaluation of *f*. Since market data are given with different strikes *K* and maturities *T*, one can substantially reduce the computational effort replacing the *M* Black-Scholes equations (15) in (21) by, whenever available, one single Dupire's equation (17). Hence, the optimization problem (21) becomes

$$\min_{\sigma\in\mathcal{H}} f(\sigma) := \frac{1}{2}\sum_{m=1}^{M}\left(D(K_m,T_m)-C_m^{obs}\right)^2 \tag{22}$$
$$\text{s.t.}\quad D=\frac{1}{2}\sigma^2(K,T)K^2D-(r-d)KD-dD,\quad (K,T)\in\Omega,$$
$$D(K,0)=\max(S-K,0),\quad K\in(0,\infty),$$

with $\Omega = (0,\infty)\times(0,T_{\max}]$. The main tasks, when solving PDE constrained optimization problems like (21) or (22), are: (i) the discretization of the model PDE, (ii) the parametrization of the parameter functions, for instance, the local volatility function $\sigma(\cdot,\cdot)$ and (iii) the type of regularization to overcome the potential ill-posedness of the problem. Finally, (iv) the efficient computation of derivative information is a crucial issue to apply standard optimization routines. A gradient evaluation at low cost accelerates the calibration procedure tremendously and makes the calibration problem amenable for the application in practice.

Since (i) is more a pricing issue, references on proper discretization schemes of the model PDE are already given in Section 2. As mentioned earlier (cf. Section 2.1), the parametrization of model functions can either be motivated by desired model properties in terms of hedging performance and implied volatility behavior or by reducing the number of unknowns in the calibration procedure. While in the time-dependent Heston model the corresponding parameter functions are usually parametrized via piecewise constant or piecewise linear functions, the type of parametrization of the local volatility function varies strongly in the financial market literature. Beaglehole and Chebanier (2002), for example, used piecewise quadratic functions, whereas Brown and Randall (1999) applied hyperbolic trigonometric functions. McIntyre (2001) considered Hermite polynomials and B-splines were used by Hamida and Cont (2005). Bicubic splines were applied to parametrize the local volatility function, for instance, in (Coleman *et al*., 1999; Jackson *et al*., 1999; Pironneau, 2009). Spline representations, however, require rectangular parametrization grids. Orosi (2010), Glover and Ali (2011) and Coleman *et al*. (2011) use radial basis functions like Gaussian, multi-quadratic and thin-plate splines, a popular choice for reconstruction surfaces from sparse data. The main advantage is the arbitrary placement of parametrization knots, which allows the number of parameters to be kept to a minimum. According to Glover and Ali (2011), thin-plate splines seem to perform best in terms of showing accurate and robust solutions to parametrize the local volatility function.

A clever parametrization strategy can be very helpful to stabilize the calibration procedure, since it reduces the number of unknowns. However, the right choice can be a difficult task. As an example, parametrizing the local volatility function with cubic splines (as in (Coleman *et al*., 1999)) is very difficult to automate, since one is faced with the trade-off between a lack of stability and unrealistic oscillations. Thus, in case of a high number of unknowns, usually some regularization is needed to overcome the ill-posedness of the optimization problem. The best-known stabilization method for ill-posed nonlinear inverse problems is the Tikhonov regularization introduced by Tikhonov (1963). Bodurtha (2000) and Bodurtha and Jermakyan (1999), for instance, minimize the sum of squared deviations from the Black-Scholes constant variance σ_0^2, i.e. $\min \| f(x,t) \|_{L^2}^2$, where $\sigma^2(x,t) = \sigma_0^2 + f(x,t)$.[10] Lagnado and Osher (1997a, 1997b), Jackson *et al*. (1999) and Coleman *et al*. (2001) suggest regularizing the ill-posed inverse problem by additionally minimizing the L^2-norm of the gradient of the volatility function, i.e. $\min \| \nabla\sigma(x,t) \|_{L^2}^2$, subject to a finite number of constraints. Theoretical stability and convergence results for the calibration problem of local volatility models can be found in (Jiang and Tao, 2001; Crépey, 2003; Egger and Engl, 2005). Egger and Engl (2005) prove convergence rates for the case of time-independent volatilities, i.e. $\sigma = \sigma(S)$, under simple and interpretable smoothness assumptions. The results can be generalized to a class of local volatility functions, where the term structure is assumed to be known, more precisely, where $\sigma(S,t) = \sigma(S)\rho(t)$ and the function $\rho(t)$ is known. Purely time-dependent volatilities are extensively studied in (Hein and Hofmann, 2003) and the ill-posedness of the inverse problem is proven. Uniqueness for state-dependent volatilities and the relation of optimal control problems corresponding to time-discrete and time-continuous observations are investigated in (Jiang and Tao, 2001).

As mentioned before, the efficient derivative evaluation is crucial when applying existing optimization routines. Coleman et al. (1999) use a trust region / interior point method and formulate the box-constrained

[10]Note that Bodurtha (2000) and Bodurtha and Jemarkyan (1999) used a trinomial tree model (cf. Section 3.1) to compute the option prices.

nonlinear least-squares problem,

$$\min_{\bar{\sigma}\in\mathcal{R}^p} f(\sigma) := \frac{1}{2}\|R(\bar{\sigma})\|_2^2 \qquad (23)$$
$$\text{s.t.} \qquad l \le \bar{\sigma} \le u \ ,$$

where $\bar{\sigma} = (\bar{\sigma}_1, \cdots, \bar{\sigma}_p)^T$ is the vector of a cubic spline parameterization of the local volatility function $\sigma(x, t)$, l and u the vectors of lower and upper bounds on $\bar{\sigma}$, respectively, and $R(\bar{\sigma})$ the residual function defined as

$$R(\bar{\sigma}) := (D(\bar{\sigma}; K_1, T_1) - C_1^{obs}, \cdots, D(\bar{\sigma}; K_M, T_M) - C_M^{obs})^T \ ,$$

where $D(\bar{\sigma};\cdot,\cdot)$ is the solution of Dupire's equation (21).[11] In (Coleman *et al*., 1999) two possibilities to efficiently compute the Jacobian of R are explored, i.e. the use of automatic differentiation (AD) (see (Griewank and Walther, 2008; Coleman and Verma, 1996)) and an approximation of the Jacobian via a secant update formula. He *et al*. (2006) extended the calibration method of Coleman *et al*. (1999) to a jump diffusion model coupled with local volatilities. When applying a classical steepest decent algorithm or quasi-Newton approach, the use of adjoint equations to compute the gradient of f at low costs is proposed in (Achdou and Pironneau, 2005; Egger and Engl, 2005; Loerx *et al*., 2010, unpublished data). An optimal control framework also using adjoints is applied by Jiang *et al*. (2003) to recover the local volatility surface. Turinici (2008) chose an SQP method to solve optimization problem (21). Note that the SQP method already needs some second order information of f which comes at a cost of solving $M+1$ Black-Scholes PDEs. According to Coleman *et al*. (1999), optimization approaches, which do not require the calculation of second order information, typically converge very slowly, such that the additional computational effort to obtain at least some second order information can be profitable in the overall computation time of the optimization routine (see also (Loerx, 2011)). Schulze (2002) developed an inexact Gauss-Newton method to recover a non-parametric local volatility function. Since, in general, the Jacobian of the residual function cannot be stored in the non-parametric setting, the Gauss-Newton subproblems are solved with an iterative method (CG method). The matrix-vector products, which are needed within the CG framework, can be provided via sensitivity and adjoint equations. This approach was further improved in terms of computational efficiency by Loerx *et al*. (2011, unpublished data). A reduced order model technique, known from fluid dynamics, is used in (Pironneau, 2009) and applied to the local volatility framework. In (Sachs and Schu, 2008, 2010) reduced order models using proper orthogonal decomposition (POD) are used to solve the PIDE problem of jump diffusion models including local volatility.

Based on Berestycki *et al*. (2002), Turinici (2008, 2009a, 2009b) discovered other forms of cost functionals, for instance, including implied volatilities and proved convergence and stability properties. Originally, Berestycki *et al*. (2002) derived a new cost functional based on results of asymptotic relations between implied and local volatility. The new functional is close to a convex functional at least for short term maturities and therefore exhibits a more stable minimizer. More precisely, Berestycki *et al*. (2002) proved that near expiry, the implied volatility can be represented as the spatial harmonic mean of the local volatility. Furthermore, they showed that, for deep in- and out-of-the-money options under certain assumptions, the squared implied volatility can be expressed as a time weighted average of squared local volatilities. These results can either be exploited to regularize the ill-posed inverse problem in a least-squares framework (as in (Turinici, 2008, 2009a, 2009b)), or it can be used to continuously extrapolate the implied volatility surface into regions, where no observed implied volatilities are available. Therefore, one is able to overcome one of the main drawbacks of extra- and interpolation approaches using Dupire's formula.

Another interesting approach, known from the field of *'dynamic programming'*, is to consider the calibration of financial market models in the framework of a *'stochastic control problem'*. The method was proposed by Avellaneda *et al*. (1997) and studied in a deeper way by Samperi (2002). In contrast to the previous methods, this approach does not rely on any parametrization of the volatility. It leads to an unconstrained

[11]Note the equivalence of calibration problem (22) and (23).

optimization problem at the cost of solving nonlinear Hamilton-Jacobi-Bellman equations. As a regularizer Avellaneda *et al*. (1997) minimize the relative-entropy distance to a prior given distribution. Some further details can also be found in (Achdou and Pironneau, 2005, Chap. 8).

4. Conclusion

To capture realistic asset price behavior and volatility dynamics, a variety of financial market models has been developed over the last 40 years. In practice smile-consistent models are used to extract this market information from frequently traded standardized options in order to price and hedge exotic derivative products. We introduced the most common models and briefly discussed their characteristics from different perspectives. From the modeling side, for instance, we saw that despite all criticism simple models are widely used in practice, due to smaller computation and implementation costs and a better intuitiveness for risk-takers.

Numerical methods are needed whenever no closed form solution is available. We discussed pros and cons of common numerical methods, like the Monte Carlo method or PDE methods. Dupire or Dupire-like equations have proven to be particularly useful for pricing and calibration purposes, since they provide option prices for different strikes and maturities with a computational effort of one PDE evaluation.

The main objective of this paper, however, was to illustrate one of the key issues in quantitative finance and that is model calibration. Thus, we reviewed the existing literature on extra- and interpolation techniques as well as iterative methods applied to analytic approximation. Whereas the previous methods are mostly restricted to the calibration of local volatility models, optimization-based calibration methods offer more flexibility and are applicable to almost any financial market model. Hence, we introduced SDE and PDE constrained optimization and addressed issues like parameter parameterization and problem regularization. The efficiency of optimization-based methods strongly depends on the computational effort necessary to compute derivative information. We emphasized that adjoint techniques for derivative computations, recently introduced in financial market literature, have the potential to substantially speed-up optimization-based calibration procedures.

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