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Projection of Polyhedra onto Coxeter Planes Described with Quaternions

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ABSTRACT: 3-dimensional convex uniform polyhedra have been projected onto their corresponding Coxeter planes defined by the simple roots of the Coxeter diagram $I_2(h)$. The projected vertices of a polyhedron onto its corresponding Coxeter plane constitute an orbit of the dihedral group D_h. In this paper, we use the quaternionic representations of the Coxeter group elements and the vertices of the polyhedra.

Keywords: Coxeter group; Group Theory; Quaternions; Projection.

إسقاط متعددات السطوح على سطح كوزتر باستخدام الرباعيات

مظاهر العجمي و مهمت كوجا وحشيمة بيت بوسلاسل

ملخص: الإسقاط العمودي للمجسمات الأفلاطونية والأرخميدية والكاتالانية ثلاثية الأبعاد على سطح كوكزتر (Coxeter plane) يتم باستخدام الجذور البسيطة لأشكال كوكزتر D_h (Coxeter diagram I_2(h)). المتجهات الساقطة لهذه المجسمات على سطح كوكزتر (Coxeter) تحتوي على التماثل النقطي للمجموعات ثنائية الأسطح (dihedral group). استخدمنا الرباعيات (quaternions) للتعبير عن عناصر مجموعة كوكزتر (Coxeter) وعن رؤوس المجسمات في هذه الورقة البحثية.

مفتاح الكلمات: مجموعة كوكزتر، نظرية المجموعات ، الرباعيات، الاسقاط.

1. Introduction

Symmetries of objects play a vital role in the analysis of structure, bonding and spectroscopy of molecules in several scientific disciplines. Group theory provides a great tool for understanding symmetry phenomena in physics. Using group theory to study the symmetry of polyhedra is helpful for physicists to develop an understanding of crystallography and quasi-crystallography. In this paper we use the rank-3 Coxeter-Dynkin diagrams, a special technique in Lie algebra, to construct the symmetries of polyhedra.

A symmetry group may contain transformations such as reflections, rotations, rotary reflections and inversions which leave the object unchanged. Coxeter groups are abstractions of the symmetry groups only generated by reflections group elements. They are useful in describing the symmetry of regular and semi-regular polytopes in arbitrary dimensions [1].

The list of convex uniform polyhedra contains Platonic solid, Archimedean solids and their duals, the Catalan solids. These special polyhedra have been successfully used to describe the crystals in physics, molecular symmetry in chemistry and some viral structures in biology [2]. To describe the symmetry of all convex uniform polyhedra we use three main finite Coxeter groups A_3, B_3, and H_3.

Invoking quaternions is an efficient tool for representing elements of the Coxeter groups of rank-3 and rank-4. Quaternions have been used to represent rotations and reflections since their discovery in 1843 [3] and have now become used to represent the Coxeter groups [4]. They are also used to symbolize the simple roots and weights of Coxeter-Dynkin diagrams. Furthermore the vertices of polyhedra are written in terms of pure imaginary quaternions. Quaternions are used because of their compactness, stability and simplicity, which make this method easy to use when compared to other methods that are based on orthogonal matrices or Euler angles.

The objective of this paper is to project the vertices of several 3D polyhedra onto their corresponding 2D Coxeter planes. We define a Coxeter plane by the simple roots of the Coxeter diagram $I_2(h)$ where h is the Coxeter number. When projected onto the Coxeter plane, the vertices of any polyhedron form orbits of the dihedral group D_h.

Orthogonal projection of higher dimensional lattices onto 2 or 3 dimensional subspaces can be related to quasi-crystal structures.

We organize the paper as follows. In section 2, definitions of finite reflection groups, quaternions, and Coxeter groups are given. We then describe the Platonic solids, the Archimedean solids and their duals with respectively their *symmetry groups* in section 3. In section 4, we study the projections of the convex uniform polyhedra. Finally, section 5 contains our conclusive remarks.

2. Finite Reflection Groups

The dihedral group D_n is the group which involves the rotation through an angle $\frac{2\pi}{n}$, which is expressible as the product of two reflections r_1 and r_2 whose mirrors have an angle of $\frac{\pi}{n}$ between them. Here, plane symmetries preserve a regular n-sided polygon where $n \geq 3$. The reflections r_1 and r_2 generate a dihedral group D_n, which is a subgroup of the symmetric group of order $2n$.

A hypercomplex number q is defined as *quaternion*, which can be written in general as

$$q = e_0q_0 + e_1q_1 + e_2q_2 + e_3q_3 = \sum_{k=0}^{3} e_kq_k \tag{1}$$

with $q_0, q_1, q_2, q_3 \in R$, R being the set of real numbers, and $e_0, e_{1,} e_{2,}$ and e_3 the quaternionic units which satisfy

$$e_ie_j = -\delta_{ij} + \varepsilon_{ij}e_k,$$

$$(i,j,k=1,2,3) \tag{2}$$

where δ_{ij} and ε_{ijk} are the Kronecker and Levi-Civita symbols, and where summation over the repeated indices is implicit.

Any quaternion can be split into two parts, a scalar part: $e_0q_0 = q_s$, and a vector part: $e_1q_1 + e_2q_2 + e_3q_3 = q_v$. The following definitions will be useful in our discussions.

The conjugate of a quaternion is defined by

$$\bar{q} = e_0q_0 - e_1q_1 - e_2q_2 - e_3q_3. \tag{3}$$

- We define the inverse quaternion q^{-1} as

$$\left(\frac{1}{|q|^2}\right)\bar{q} = qq^{-1} = q^{-1}q = e_{0.} \tag{4}$$

- The scalar product of two quaternions q and p is given by

$$(q,p) = q_0p_0 + q_1p_1 + q_2p_2 + q_3p_3, \tag{5}$$

or as

$$(q,p) = \frac{1}{2}(\bar{q}p + \bar{p}q\ ). \tag{6}$$

- The product of q and $\bar{q}$ gives a real non-negative number called the norm of q .

$$q\bar{q} = (e_0, e_1, e_3, e_3)(q_0^2 + q_1^2 + q_2^2 + q_3^2, 0,0,0)^T$$

$$= \sum_{k=o}^{3} q_k^2 = |q|^2 = \bar{q}q \tag{7}$$

The norm of q does not equal zero and is the length of the quaternion. The unit quaternion is a normalized quaternion when $|q| = 1$.

- The binary tetrahedral *(BT)* group T is represented by the following set of 24 quaternions [5].

$$T = \left\{\pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)\right\} \tag{8}$$

This set represents the vertices of the 24-cell.

- The binary octahedral (BO) group consists of the union of set T and T' [3] where

$$T' = \begin{Bmatrix} \frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3), \frac{1}{\sqrt{2}}(\pm 1 \pm e_2), \\ \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1), \frac{1}{\sqrt{2}}(\pm 1 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2) \end{Bmatrix}. \tag{9}$$

The order of the group

$$O = T \oplus T' \tag{10}$$

is 48.

- The binary icosahedral (BI) group is of order 60×2=120. The union of two sets T and S represents the set of quaternions of the binary icosahedral group I as

$$I = T \oplus S. \tag{11}$$

The set S in the quaterionic form is [6]:

$$\begin{aligned} S = {} & \tfrac{1}{2}(\pm\tau \pm e_1 \pm \sigma e_3), \tfrac{1}{2}(\pm\tau \pm e_2 \pm \sigma e_1), \tfrac{1}{2}(\pm\tau \pm e_3 \pm \sigma e_2), \\ & \tfrac{1}{2}(\pm\sigma \pm e_1 \pm \tau e_3), \tfrac{1}{2}(\pm\sigma \pm e_2 \pm \tau e_1), \tfrac{1}{2}(\pm\sigma \pm e_3 \pm \tau e_2), \\ & \tfrac{1}{2}(\pm 1 \pm \tau e_1 \pm \sigma e_2), \tfrac{1}{2}(\pm 1 \pm \tau e_2 \pm \sigma e_3), \tfrac{1}{2}(\pm 1 \pm \tau e_3 \pm \sigma e_1), \\ & \tfrac{1}{2}(\pm\sigma e_1 \pm \tau e_2 \pm e_3), \tfrac{1}{2}(\pm\sigma e_2 \pm \tau e_3 \pm e_1), \\ & \tfrac{1}{2}(\pm\sigma e_3 \pm \tau e_1 \pm e_2). \end{aligned} \tag{12}$$

where $\tau = \frac{1+\sqrt{5}}{2}, \sigma = \frac{1-\sqrt{5}}{2}$ are the golden ratio and its algebraic conjugate respectively.

The vertices of the 600-cell [7] consists of the set of quaternions I. However, the set S represents the vertices of the snub 24-cell [8].

A Coxeter group W is defined as an abstract group that is generated by elements $R = \{r_i, i = 1,2, \ldots n\}$ subject to the relations

$$(r_i r_j)^{m_{ij}} = 1 \quad \text{for all } i, j \in [n] \tag{13}$$

Where $m_{ii} = 1$ and $m_{ij} = m_{ji} > 1$ for all $, j \in \{1, \ldots, n\}$, so that $i \neq j$. The pair (W, R) is called a Coxeter system; it consists of a group W and a set of generators R.

The Cartan matrix is defined by

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}. \tag{14}$$

The matrix elements are scalar products of two simple roots and satisfy the following criteria:

1. Every Cartan matrix has an inverse defined by

$$(C^{-1})_{ij} = (\omega_i, \omega_j) \tag{15}$$

where ω is the fundamental weight defined as the *basis vectors of the dual space*. The real and the dual space are in the same space.

2. The relation between ω and α satisfies:

$$(\omega_i, \alpha_j) = \delta_{ij} \tag{16}$$

$$\alpha_i = C_{ij}\omega_j \tag{17}$$

$$\omega_i = C_{ij}^{-1}\alpha_j. \tag{18}$$

The summation over repeated indices is implicit.

Finite Coxeter groups are generated by simple reflections as finite Euclidean reflection groups.
A reflection of a vector Λ in a finite dimensional Euclidean space is a linear transformation by a reflection generator r of the finite Coxeter group with respect to a hyperplane perpendicular to the root vector α. This can be defined by the formula [5]

$$r\Lambda = \Lambda - \frac{2\,(\Lambda,\alpha)}{(\alpha\,,\alpha)}\,\alpha. \tag{19}$$

Here, α is a vector which is usually called the root vector or simply root. This root vector is called *a simple root* in Lie algebra. The norm of the simple root is fixed as $\sqrt{2}$.

If α is an arbitrary simple root written in terms of quaternions, then the reflection of an arbitrary vector Λ with respect to the plane orthogonal to the simple root α can be written as the product of quaternions

$$r_i\Lambda = -\frac{\alpha_i}{\sqrt{2}}\overline{\Lambda}\frac{\alpha_i}{\sqrt{2}}. \tag{20}$$

We then define this reflection operation by a different notation as

$$r_i\Lambda = \left[-\frac{\alpha_i}{\sqrt{2}}\,,\frac{\alpha_i}{\sqrt{2}}\right]^{*}\Lambda. \tag{21}$$

Here, the asterisk signifies that we take the conjugate of Λ.
However, a rotation can be written in our notation as

$$\Lambda \to [\alpha, \bar{\alpha}]\Lambda. \tag{22}$$

3. Polyhedra

Platonic and Archimedean Solids

Platonic solids are made of regular polygons (triangle, square or pentagon) meeting in identical vertices. On the other hand, the Archimedean solids are convex polyhedra whose faces are regular polygons. One can obtain 5 of them by truncation of the Platonic solids and the others by expansion and snubification. The Archimedean solids are vertex-transitive (vertices move to each other in any symmetry operation). The method explained in [4] gives only the vertices of 11 Archimedean solids. The other two chiral solid vertices are taken from [9]. The first Platonic solid is the tetrahedron and its polyhedral group is the *tetrahedral group* $W(A_3)$ where its group elements can be written as

$$W(A_3) = \{[p,\bar{p}] \oplus [t,\bar{t}]^{*}\};\; p \in T,\; t \in T'. \tag{23}$$

The Coxeter-Weyl group $W(A_3)$ is represented by the Coxeter-Dynkin diagram shown in Figure 1 [6]

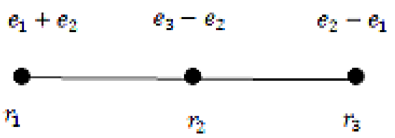


Figure 1. Coxeter-Dynkin diagram of $W(A_3)$.

To obtain the orbit of the group, corresponding to the vertices of the tetrahedron, we use the method described in [6]. We apply the group elements on the highest weight written in a general form: $\Lambda = (a_1, a_2, a_3) = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$ which can be written in terms of quaternionic imaginary units as

$$\Lambda = \alpha \mathrm{e}_1 + \beta \mathrm{e}_2 + \gamma \mathrm{e}_3.$$

In [6], the values of α, β and γ are found from eq (18). Replacing these values in Λ, we obtain:

$$\alpha = \frac{1}{2}(a_1 - a_3), \beta = \frac{1}{2}(a_1 + a_3)\,,$$
$$\gamma = \frac{1}{2}(a_1 + 2a_2 + a_3).$$

The seven special cases of (a_1, a_2, a_3) are indicated by ones or zeros and called the Dynkin indices.
The cube and the octahedron are dual of each other, and they also have the same polyhedral symmetry group, which is the *octahedral group* $W(B_3)$ with 48 elements given by

$$\begin{aligned}&W(B_3) \approx Aut(A_3) \approx S_4 \times C_2 =\\ &\{[p,\bar{p}] \oplus [p,\bar{p}]^{*} \oplus [t,\bar{t}] \oplus [t,\bar{t}]^{*}\};\\ &p \in T, t \in T'.\end{aligned} \tag{24}$$

The Coxeter group of this polyhedral symmetry group is represented by the Coxeter-Dynkin diagram shown in Figure 2.

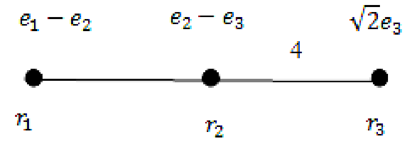


Figure 2. Coxeter-Dynkin Diagram of $W(B_3)$.

The other Platonic solids that are dual of each other are the icosahedron and the dodecahedron. Their polyhedral symmetry group is the *icosahedral group* $W(H_3)$ of order 120 given by

$$W(H_3) \approx A_5 \times C_2 = \{[p, \bar{p}] \oplus [p, \bar{p}]^*\}; p \in I \tag{25}$$

The Coxeter-Dynkin diagram of the icosahedral group is shown in Figure 3.

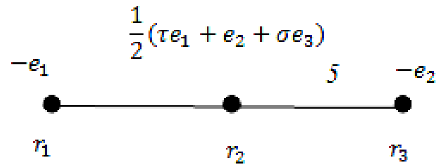


Figure 3. Coxeter-Dynkin diagram of $W(H_3)$.

The five different Platonic solids are displayed in Table 1.

Catalan Solids

To construct the dual of a solid we connect vertices which go through the centers of the faces of each solid. The set of new polyhedra is called the *Catalan solids.* They are not semi-regular since they have more than one set of vertices and their faces are not regular polygons in general. They are made of scalene triangles, isosceles triangles, rhombuses, kites or irregular pentagons. Obviously they are still convex.

The symmetry groups of Catalan solids are the same as their Archimedean duals. Some of the Archimedean and Catalan solids do not have the full symmetry group of a Platonic solid but only the rotational symmetry group of the octahedron or the icosahedron.

The Catalan solids are face-transitive (faces move onto each other in any symmetry operation). In this paper, we take vertices of the Catalan solid from [7].

The Archimedean and their dual Catalan solids are displayed in Tables 1-4.

4. Projecting Polyhedra onto Their Coxeter Planes

Coxeter plane

The product of all reflections in a Coxeter group is called the Coxeter element: $c = r_1 r_2 \ldots$ and its order is called Coxeter number h. There is a unique plane P on which the Coxeter element acts as a rotation by an angle $\frac{2\pi}{h}$. This plane is called the Coxeter plane [1]. Every Coxeter group has a maximal subgroup isomorphic to the dihedral group of order $2h$. We use the following technique to determine the Coxeter planes of the three main polyhedaral symmetry groups.

The Coxeter group $W(A_3)$ elements are given in eq. (8) and the group is represented by the Coxeter-Dynkin diagram shown in Figure 1. The Coxeter element of this group is $c = r_1 r_2 r_3$ where its order (Coxeter number) $h = 4$ since $(r_1 r_2 r_3)^4 = 1$. Let us define a dihedral group D_h as a Coxeter plane with generators R_1 and R_2 of $W(A_3)$. We define R_1 as the product of generators r_1 and r_3 which are orthogonal to each other

$$R_1 = r_1 r_3 = r_3 r_1 \tag{26}$$

and the generator R_2 as

$$R_2 = r_2. \tag{27}$$

The dihedral group $D_h = \langle R_1, R_2 \rangle$ is a subgroup of $W(A_3)$ where $R_1^2 = R_2^2 = (R_1 R_2)^4 = 1$. The dihedral group is D_4 and has 8 elements. The dihedral group D_4 represented by the Coxeter-Dynkin diagram is shown in Figure 4,

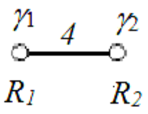


Figure 4. Coxeter-Dynkin diagram of $\boldsymbol{I_2(4)}$.

where γ_1 and γ_2 are the simple roots and $R_1^2 = R_2^2 = (R_1R_2)^4 = 1$.

The norm and the scalar products are given as $\|\gamma_1\| = \|\gamma_2\| = \sqrt{2}$,

$$(\gamma_1, \gamma_2) = 2\cos\left(\pi - \frac{\pi}{4}\right) = -\sqrt{2}. \tag{28}$$

Using eq. (14) the Cartan matrix of D_4 is $C_{D_4} = \begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$.

Using eq. (19), the generators R_1 and R_2 of D_4 correspond to a reflection at a hyperplane perpendicular to the root vectors γ's given as

$$R_1\gamma_1 = -\gamma_1 \tag{29}$$

$$R_2\gamma_2 = -\gamma_2 \tag{30}$$

$$R_1\gamma_2 = \gamma_2 + \sqrt{2}\gamma_1 \tag{31}$$

$$R_2\gamma_1 = \gamma_1 + \sqrt{2}\gamma_2 \ . \tag{32}$$

In eqs (31 & 32) we use eq. (19), replacing α with γ. In eq. (29) we define γ_1 as a linear combination of α_1 and α_3:

$$\gamma_1 = x_1\alpha_1 + x_3\alpha_3 \tag{33}$$

where x_2 equals zero, and in eq (30), γ_2 is defined as

$$\gamma_2 = x_2\alpha_2 \tag{34}$$

where x_2 and x_3 equal zero. Combining the equations above we can get the values of coefficients x_1, x_2 and x_3:

$x_1 = x_3 = \frac{1}{\sqrt{2}}$,

$x_2 = 1$.

Then, eqs (33 & 34) become

$\gamma_1 = \sqrt{2}\, e_2$

$\gamma_2 = e_3 - e_2$

These simple roots γ_1 and γ_2 define the Coxeter plane. However γ_1 and γ_2 are not orthogonal to each other. So, we define an orthogonal set of vertices given by

$$\hat{\imath} = \frac{\gamma_1}{\sqrt{2}} = e_2 \tag{35}$$

$$\hat{\jmath} = e_3 \tag{36}$$

where $\hat{\imath}$ in x-axis , $\hat{\jmath}$ in y-axis and $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = 1$.

Using the same argument as above, we define the Coxeter plane as D_6=$\langle R_1R_2\rangle$ of $W(B_3)$ with generators R_1 and R_2 where the Coxeter number of B_3 equals 6. The dihedral group D_6 represented by the Coxeter-Dynkin diagram is shown in Figure 5.

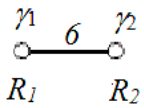


Figure 5. Coxeter-Dynkin diagram of $\boldsymbol{I_2(6)}$.

γ_1 and γ_2 are the simple roots and $R_1^2 = R_2^2 = (R_1R_2)^6 = 1$.

Using eq. (14) the Cartan matrix of D_6 is $C_{D_6} = \begin{pmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix}$

Using the method explained above for the $I_2(4)$, γ_1 and γ_2 for the $I_2(6)$ can be written in terms of quaternions as

$$\gamma_1 = \frac{1}{\sqrt{3}}(e_1 - e_2 + 2e_3) \tag{37}$$

$$\gamma_2 = e_2 - e_3 \tag{38}$$

where $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, and $\alpha_3 = \sqrt{2}e_3$ [6]. The two components of γ_1 and γ_2 in the x-axis and y-axis are orthogonal to each other:

$$\hat{\imath} = \frac{\gamma_1}{\sqrt{2}} = \frac{1}{\sqrt{6}}(e_1 - e_2 + 2e_3) \tag{39}$$

$$\hat{\jmath} = \frac{1}{\sqrt{2}}(e_1 + e_2) \tag{40}$$

where $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = 1$.

Using the argument introduced above, we define the Coxeter plane as $D_{10} = \langle R_1R_2 \rangle$ of $W(H_3)$ with generators R_1 and R_2 respectively. The Cartan matrix of D_{10} is

$$C_{D_{10}} = \begin{pmatrix} 2 & -\sqrt{\tau+2} \\ -\sqrt{\tau+2} & 2 \end{pmatrix}$$

The coefficients x_1 and x_3 are

$$x_1 = \frac{\tau}{\sqrt{\tau+2}}$$

$$x_3 = \frac{1}{\sqrt{\tau+2}}$$

and the coefficient $x_2 = 1$. Then γ_1 and γ_2 can be written in terms of quaternions as $\gamma_1 = \frac{\sqrt{2}}{c}(\tau e_1 + e_2)$,

$$c = \sqrt{\tau+2} \tag{41}$$

$$\gamma_2 = \frac{1}{\sqrt{2}}(\tau e_1 + e_2 + \sigma e_3) \tag{42}$$

From [4], $\alpha_1 = -e_1$, $\alpha_2 = \frac{1}{2}(\tau e_1 + e_2 + \sigma e_3)$, and $\alpha_3 = -e_2$. The two orthogonal components of γ_1 and γ_2 are $\hat{\imath}$ in x -axis and $\hat{\jmath}$ in y -axis

$$\hat{\imath} = \frac{\gamma_1}{\sqrt{2}} = \frac{1}{c}(\tau e_1 + e_2) \tag{43}$$

$$\hat{\jmath} = e_3. \tag{44}$$

Projection

In our work we use the orthogonal projection of polyhedra in three-dimensions onto their corresponding Coxeter planes. By taking the scalar product of an arbitrary vector $\wedge$ of a polyhedron with $\hat{\imath}$ and $\hat{\jmath}$ of the Coxeter plane for $W(A_3)$, $W(B_3)$ and $W(H_3)$, we can determine the horizontal and vertical components of the projected vectors P_x and P_y of the orbit $O(a_1\ a_2\ a_3)$ onto the Coxeter plane.

Projection of polyhedra having tetrahedral symmetry.

We can take the scalar product of the vertices of a polyhedron with $\hat{\imath}$ and $\hat{\jmath}$ in eqs (31 & 32) to get P_x and P_y of vertices of the polyhedra having tetrahedral symmetry. Let us now consider the tetrahedron which has tetrahedral symmetry. Here, $\Lambda = (100)$ or $\Lambda = (001)$. When the vertex $\frac{1}{2}(e_1 + e_2 + e_3)$ is multiplied (as a scalar product) with $\hat{\imath}$ and $\hat{\jmath}$, the horizontal and vertical projections become respectively

$$P_x = \left(e_2, \frac{1}{2}(e_1 + e_2 + e_3)\right) = \frac{1}{2}$$

$$P_y = \left(e_3, \frac{1}{2}(e_1 + e_2 + e_3)\right) = \frac{1}{2}$$

When this projection is applied to all vertices of $\Lambda = (100)$, we obtain the full projection of the tetrahedron in Figure 6. The solid line shows the projection of the front faces and the dashed line represents the projection of the back faces, and this applies to all following figures.

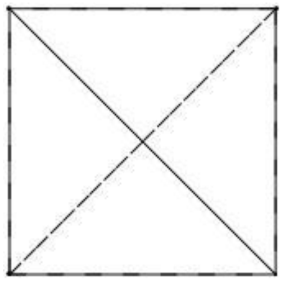


Figure 6. Projection of the tetrahedron.

The radius in this projection equals $\frac{1}{\sqrt{2}}$. The angle between all vertices of this polyhedron and the x-axis is 45°. The lines between the vertices indicate the nearest neighbor pairs. We find the same result for the vector $\Lambda = (001)$. In the figure above it is clear that there are 4 vertices at the corners with but no central vertex with 6 edges, and that the faces of the tetrahedron are triangular. The angle between the edges is 90°. Therefore this polygon has 4-fold rotational symmetry. In the case of truncated tetrahedron $\Lambda = (011)$ or $\Lambda = (110)$, the projected vectors result in two circles. The first one has a radius of $\frac{1}{\sqrt{2}}$ with 4 projected vertices each having an angle of 45° with the x-axis. The second one has a radius of $\sqrt{\frac{5}{2}}$ with 8 projected vertices of $\frac{1}{2}(\pm e_2 \pm 3e_3)$ and $\frac{1}{2}(\pm 3e_2 \pm e_3)$. The projected vertices of $\frac{1}{2}(\pm e_2 \pm 3e_3)$ and $\frac{1}{2}(\pm 3e_2 \pm e_3)$ have angles equal to 71.56° and 18.4° with the *x*-axis respectively as shown in Figure 7.

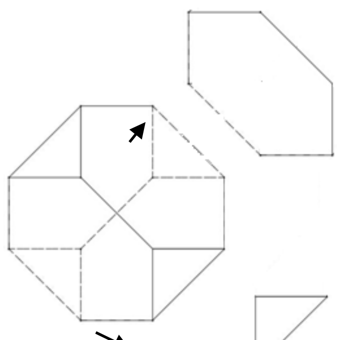


Figure 7. Projection of the truncated tetrahedron.

The projection shows us 18 edges of the truncated tetrahedron with 8 faces (4 triangles and 4 hexagons) and 12 vertices. This projection an “edge first” projection with 8-fold symmetry.
We can apply a similar projection to the dual of the truncated tetrahedron which has a $W(A_3)$ symmetry group. The projected vertices are shown Figure 8.

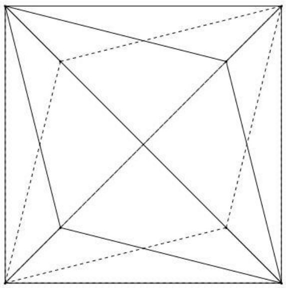


Figure 8. "Edge first" projection of the dual of the truncated tetrahedron.

Applying the same projection to the other vertices of tetrahedral solids we can get their projections. Table 1 lists the results of projecting the Platonic solids onto their Coxeter planes. Table 2 lists the results of projecting solids having $W(A_3)$ symmetry onto their Coxeter planes.

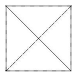
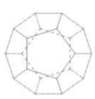
We take the scalar product of the vertices of polyhedra that have the octahedral symmetry with $\hat{\imath}$ and $\hat{\jmath}$ of $W(B_3)$ as defined in eqs. (39 & 40) to get their P_x and P_y components in the Coxeter plane. Table 3 lists the results of the projections of the Archimedean solids onto their Coxeter plane.

Taking the scalar product of the vertices of each polyhedron having an icosahedral symmetry with $\hat{\imath}$ and $\hat{\jmath}$ in eqs. (43 & 44) we get their P_x and P_y onto the (*x*,*y*)-plane of $W(H_3)$. The set of invariant points of the polyhedra having icosahedral symmetry projected onto the Coxeter plane of $W(H_3)$ is listed in Table 3.

5. Conclusion

In this paper we projected the Platonic, regular Archimedean and Catalan polyhedra in $3D$ onto the Coxeter plane. We defined the Coxeter plane by the root system of the dihedral group generated by Coxeter-Dynkin diagram D_n. Our projection preserves the h-fold symmetry of the root system which is a subgroup of its complete symmetry. Obviously, after projection, the regular polygons become irregular. However, it can be seen how the polygons still meet at the same vertices. The Dynkin indices chosen were 0's and 1's for a general vector. This work can be extended for those polyhedra whose vertices can be constructed by choosing arbitrary integer values of Dynkin indices.

Table 1. List of the Platonic solids and their projection onto Coxeter plane.

Platonic solids	Dynkin indices of Λ	Symmetry Group	3D polyhedra	Projection	# vertices/ radius	radius
Tetrahedron	(100)	$W(A_3)$			(0,4)	0 $\frac{1}{\sqrt{2}}$
Octahedron	(100)	$W(B_3)$			(0,6)	0 $\frac{2}{\sqrt{3}}$
Cube	(001)	$W(B_3)$			(2,6)	0 $\frac{2}{\sqrt{3}}$
Dodecahedron	σ(100)	$W(H_3)$			(0,10,10)	0 $\frac{\sqrt{\tau}}{5^{1/4}}$ $\sqrt{1+\frac{2}{\sqrt{5}}}$

Platonic solids	Dynkin indices of Λ	Symmetry Group	3D polyhedra	Projection	# vertices/ radius	radius
Icosahedrons	$\sigma(001)$	$W(H_3)$			(2,10)	0 $\frac{\sqrt{\tau}}{5^{1/4}}$

Table 2. An Archimedean polyhedron having *W(A₃)* symmetry and its projection onto its Coxeter plane.

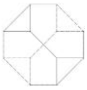
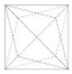
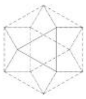
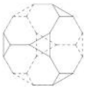
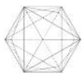
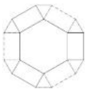
Polyhedron	3D polyhedron	Projection	# vertices/ radius	Radius	Dual	Projection	# vertices/ radius	radius
Truncated tetrahedron			(0,4,8)	$0, \frac{1}{\sqrt{2}}, \sqrt{\frac{5}{2}}$			(0,4,4)	$0, \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}$

Table 3. List of the polyhedra (Archimedean & Catalan) having *W(B₃)* symmetry and their projection onto Coxeter plane.

Polyhedra ($W(B_3)$)	3D polyhedra	Projection	# vertices / radius	Radius	Dual	Projection	# vertices / radius	radius
Cub-octahedron			(0,6,6)	0 $\sqrt{\frac{2}{3}}$ $\sqrt{2}$			(2,12)	0 $\sqrt{\frac{2}{3}}$
Truncated cube			(0,6,6,12)	0 $\sqrt{\frac{2}{3}}$ $\frac{2+\sqrt{2}}{\sqrt{3}}$ $\sqrt{\frac{10}{3}+2\sqrt{2}}$			(2,6,6)	0 $\sqrt{\frac{2}{3}}$ $2\sqrt{2-\frac{4\sqrt{2}}{3}}$
Truncated octahedron			(0,12, 12)	0 $\sqrt{2}$ $\sqrt{\frac{14}{3}}$			(2,6,6)	0 1 $2\sqrt{\frac{2}{3}}$

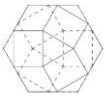
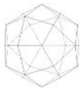
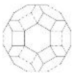
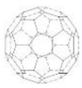
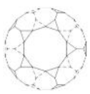
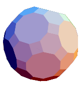
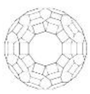
Polyhedra (W(B$_3$))	3D polyhedra	Projection	# vertices / radius	Radius	Dual	Projection	# vertices / radius	radius
Small rhombi-cubocta-hedron			(0,6, 12,6)	0 $\sqrt{\frac{2}{3}}$ $\sqrt{\frac{2}{3}(3+\sqrt{2})}$ $\frac{2+\sqrt{2}}{\sqrt{3}}$			(2,6,6,6,6)	0 $\sqrt{\frac{2}{3}}$ $\frac{2}{7}\sqrt{6-\frac{8\sqrt{2}}{3}}$ $\frac{1}{\sqrt{3}}$ 1
Great rhombicub-octa hedron			(0,12, 12,12, 12)	0 $\sqrt{2}$ $\sqrt{\frac{10}{3}+2\sqrt{2}}$ $\sqrt{6+\frac{8\sqrt{2}}{3}}$ $\sqrt{6+\frac{10\sqrt{2}}{3}}$			(2,6,6,12)	0 $\sqrt{\frac{2}{3}}$ $\frac{2}{3}\sqrt{\frac{22}{3}-4\sqrt{2}}$ $\frac{1}{7}\sqrt{17-\frac{10\sqrt{2}}{3}}$
Snub cube			(0,6,6, 6, 6)	0 $\frac{2(-2+y)\sqrt{6(7+2y+4y^2)}}{-9+18y}$ $\sqrt{\frac{2}{9}(-2+y)^2+\frac{6(8+y+2y^2)^2}{(9-18y)^2}}$ $\sqrt{\frac{2}{9}(1+y)^2+\frac{1}{54}(4-2y+\frac{18}{-1+2y})^2}$ $\frac{2\sqrt{\frac{2}{3}(-7-6\sqrt{33}+15y^2+116y)}}{-3+6y}$ where $y=(19+3\sqrt{33})^{1/3}$			(2,6,6,6 6,6)	0 $\frac{-1+2y}{3\sqrt{3}}$ $\frac{2}{(1-2y)^2\sqrt{\frac{3}{-349-60\sqrt{33}+6}}}$ $\frac{2}{\sqrt{3}}$ $\frac{2}{(1-2y)^2\sqrt{\frac{3}{95+12\sqrt{33}+42y}}}$ $\frac{4\sqrt{\frac{1}{3}(1+y)(26+3\sqrt{33}+6y)}}{(1-2y)^2}$ where $y=(19+3\sqrt{33})^{1/3}$

Table 4. List of the polyhedra (Archimedean & Catalan) having *W(H$_3$)* symmetry and their projection onto Coxeter plane.

Polyhedra (W(H$_3$))	3D polyhedra	Projection	# vertices/ radius	Radius	Dual	Projection	# vertices/ radius	Radius
Icosi- dodeca- hedron			(0,10, 10, 10)	0 $\frac{\sqrt{\tau}}{5^{1/4}}$ $\sqrt{1+\frac{2}{\sqrt{5}}}$ τ			(2,12, 10)	0 $\sqrt{\frac{1}{10}(5+\sqrt{5})}$ $\sqrt{\frac{2}{5+\sqrt{5}}}$
Truncated icosahe- dron			(0,10, 10, 20,20)	0 $\frac{\sqrt{\tau}}{5^{1/4}}$ $\sqrt{2+\frac{2}{\sqrt{5}}}$ $\sqrt{\frac{5}{2}+\frac{9}{2\sqrt{5}}}$ $\sqrt{\frac{7}{2}+\frac{11}{2\sqrt{5}}}$			(2,10, 10,10)	0 $\sqrt{\frac{1}{10}(5+\sqrt{5})}$ $\sqrt{\frac{2}{5+\sqrt{5}}}$ $\frac{3}{19}\sqrt{13+\frac{22}{\sqrt{5}}}$
Truncated dodeca- hedron			(0,20, 10, 10,20)	0 τ $\sqrt{\frac{5}{2}+\frac{11}{2\sqrt{5}}}$ $2\sqrt{1+\frac{2}{\sqrt{5}}}$ $\sqrt{\frac{9}{2}+\frac{19}{\sqrt{5}}}$			(2,10, 10)	$\sqrt{\frac{2}{5+\sqrt{5}}}$ $\frac{1}{11}\sqrt{25+2\sqrt{5}}$ $\sqrt{\frac{\frac{1}{2}(5+\sqrt{4})}{4+\sqrt{5}}}$
Small rhomb- icosi- dodecahe- dron			(0,10, 20, 20,10)	0 $\frac{\sqrt{\tau}}{5^{1/4}}$ τ $\sqrt{\frac{5}{2}+\frac{9}{2\sqrt{5}}}$			(2,10, 10, 10,10, 10,10)	0 $\sqrt{\frac{1}{10}(5+\sqrt{5})}$ $\sqrt{\frac{2}{5+\sqrt{5}}}$ $\frac{1}{11}\sqrt{26+\frac{38}{\sqrt{5}}}$

Polyhedra (W(H_3))	3D polyhedra	Projection	# vertices/ radius	Radius	Dual	Projection	# vertices/ radius	Radius
				$\sqrt{\frac{5}{2}+\frac{11}{2\sqrt{5}}}$				$\frac{1}{11}\sqrt{58+\frac{122}{\sqrt{5}}}$ $\sqrt{\frac{4}{9}+\frac{8}{9\sqrt{5}}}$ 1
Great rhomb-icosi-dodeca-hedron			(0,20, 20, 20,20, 20,20)	0 τ $\sqrt{\frac{7}{2}+\frac{11}{2\sqrt{5}}}$ $\sqrt{\frac{9}{2}+\frac{19}{\sqrt{5}}}$ $\sqrt{\frac{1}{2}(13+5\sqrt{5})}$ $\sqrt{\frac{1}{10}(75+29\sqrt{5})}$ $\sqrt{\frac{1}{10}(75+31\sqrt{5})}$			(2,10, 10, 10,10, 20)	$\sqrt{\frac{1}{10}(5+\sqrt{5})}$ $\sqrt{\frac{2}{5+\sqrt{5}}}$ $\frac{2}{5}\sqrt{5+\frac{2}{\sqrt{5}}}$ $\frac{1}{3}\sqrt{58+\frac{122}{\sqrt{5}}}$ $\frac{1}{3}\sqrt{26+\frac{38}{\sqrt{5}}}$ 1

Polyhedra (W(H_3))	3D polyhedra	Projection	# vertices/ radius	Radius	Dual	Projection	# vertices/ radius	Radius
Snub dodocahedron			(10,10, 10,10, 10,10)	$\sqrt{\frac{(1+x-x^2)^2}{\tau\sqrt{5}}+(1-\tau x^3)^2}$ $\sqrt{\frac{(-1+x+x^2)^2}{\tau\sqrt{5}}+(1-\tau x^3)^2}$ $\sqrt{x^2+\frac{\left(-2+x^2(2\sigma+2\tau^2 x)\right)^2}{4\tau\sqrt{5}}}$ $\sqrt{x^2+\frac{(-2\sqrt{5}+x^2(-2\sigma+2\tau^2 x))^2}{4\tau\sqrt{5}}]}$ $\sqrt{-4\tau^2(-1+x^2)^2+\frac{(-2+2\tau x(-1+}{\tau\sqrt{5}}}$ $x \approx 1.94315$			(2,10, 10, 10,10, 20,10, 10,10)	0 $\frac{1}{151}\sqrt{\frac{9699}{2}+\frac{10249}{\sqrt{5}}}$ $\frac{1}{302}\sqrt{26669+\frac{58871}{\sqrt{5}}}$ $\frac{1}{302}\sqrt{30043+\frac{64751}{\sqrt{5}}}$ $\frac{1}{302}\sqrt{36485+\frac{80519}{\sqrt{5}}}$ $\sqrt{\frac{615+272\sqrt{5}}{1510}}$ $\sqrt{1+\frac{1}{\sqrt{5}}}$ $\frac{3}{19}\sqrt{61+\frac{131}{\sqrt{5}}}$ $\sqrt{2+\frac{4}{\sqrt{5}}}$

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