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Article

Development of the Theory of the Functions of Real Variables in the First Decades of the Twentieth Century

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Abstract:

In (Biacino 2018) the evolution of the concept of a real function of a real variable at the beginning of the twentieth century is outlined, reporting the discussions and the polemics, in which some young French mathematicians of those years as Baire, Borel and Lebesgue were involved, about what had to be considered a genuine real function. In this paper a technical survey of the arising function and measure theory is given with a particular regard to the contribution of the Italian mathematicians Vitali, Beppo Levi, Fubini, Severini, Tonelli etc ... and also with the purpose of exposing the intermediate steps before the final formulation of Radon-Nicodým-Lebesgue's Theorem and the Italian method of calculus of variations.

Keywords: Borel and Lebesgue measurable functions; Lebesgue-Vitali's theorem; absolutely continuous functions; Vitali-Lusin's Theorem; bounded variation functions; additive set functions and their derivatives

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Introduction

In (Biacino 2018) the evolution of the concept of a real function at the beginning of the 20th century, after the introduction in the mathematical world of the Dirichlet's concept of function, the appearance of a series of pathological functions and finally the definitions of new classes of sets and functions given by the French mathematicians Baire (1874-1932), Borel (1871-1956) and Lebesgue (1875-1941), is examined. Their purposes and attempts to propose a large class of functions and sets as accessible objects, where all the mathematical classical operations could be performed, their discussions and polemics, in which many other mathematicians were involved, are reported.

In this paper the early development of measure and function theory is exposed and discussed in detail, with a particular emphasis on the new technical results about functions and the comparison of their properties. The contribution of some Italian scholars to the just

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born theory of measurable functions is underlined also: as I observed in (Biacino 2019), a particular interest in the theoretical questions about real functions and a noteworthy sensibility towards Riemann's integral theory were already present in the thought of the Italian mathematician U. Dini (1845-1918); this is why the theses of Baire, in 1898, and Lebesgue in 1902, were published on the *Annali di Matematica*, directed by Dini; indeed, the important mathematician had already developed in his famous treatise: *Fondamenti per la teorica delle funzioni di variabili reali* in 1878 noteworthy attempts at the study of the functions of a real variable in a very general setting; there he gives the notion of indetermination limits in a point, on the left and on the right, for oscillating functions and the consequent definition of derived numbers, the so called Dini derivatives, the proof of the Hankel condensation principle and a great amount of fundamental theorems, we will often quote in this paper.

Many of his disciples also gave this theory fundamental contributions. The young V. Volterra (1860-1940) in a paper published on the *Giornale di Battaglini*, in 1881 proposed the first example of a bounded derivative that is not Riemann integrable (Volterra 1881 b).

This function is a pointwise discontinuous function, that is, as in the definition in (Dini 1878, 62), even it has infinitely many discontinuities, in every part of the interval where it is defined there is at least a point of continuity for it. But it is not Riemann integrable: Dini claims that if a function is Riemann integrable then it is pointwise discontinuous, but he says also that he believed that the converse in general does not hold (Dini 1878, 250). The previous example by Volterra proves that he was right; and that pointwise discontinuous functions exist whose set of discontinuities has positive Lebesgue measure.

But mainly it proved that Riemann's theory of integral was inadequate to solve the problem of the research of the primitives of a given bounded function and justified the introduction of Lebesgue's integral, as Lebesgue himself notices in his *Leçons*. No wonder if a good series of results in measure and real function theory was produced by the successive generation, that is the generation of C. Severini (1872-1951), G. Vitali (1875-1932), F. Fubini (1879-1943), B. Levi (1875-1961) and L. Tonelli (1885-1946): the work of all these mathematicians is amply quoted and expounded in this paper, the origin of their ideas and the connections with the French scholars are also underlined. So in the following essay you will find a brief history of many important theorems about the new classes of functions introduced at the beginning of 20th century, as Lebesgue-Vitali's Theorem, Vitali characterization of integral functions, Vitali-Lusin' Theorem etc. up to Radom-Nicodým-Lebesgue's Theorem: indeed these theorems not only allowed a complete organization of the material and a significant progress of the theory but represented even a key to win the resistance of the classical mathematicians who thought that only very regular functions were useful when doing mathematics. At the same time the way was beginning to open to the important applications to the variational calculus and functional analysis.

Vitali-Lebesgue's Theorem

At the beginning of 1900, the mathematicians work mainly using analytic functions or functions provided of one or more derivatives, except at most for a finite number of points. Now in general a Riemann integrable function, a Baire function, a Lebesgue measurable function are not of such a kind. This is why a comparison of the new classes of functions becomes essential and proceeds rapidly at the same rate of growth of measure and integration theory.

Another important problem was to characterize a Riemann integrable function by means of some property of the set of its discontinuities. The idea of Hankel that a function is Riemann integrable if and only if it is pointwise continuous was proved to be wrong, as we

have said, so the problem kept unsolved. The young Vitali² in the first years of 1900 was a teacher at Sassari, Voghera and Genua. Despite he came from the school of Dini and Arzelà, he worked in general without contacts with other mathematicians³; after his important papers about analytic functions linked to the Arzelà's research, he was interested, perhaps indirectly influenced by Dini, in the works of Cantor, Borel, Jordan.

So in 1904 Vitali in his brief paper *Sulla integrabilità delle funzioni* (Vitali 1904) and Lebesgue in his *Leçons* (Lebesgue 1904 a), separately characterize Riemann integrable functions; they establish in this way a link between the rising theory of functions and classical mathematics, solving definitely a problem in which the mathematicians were interested since 1867.

The demonstrations by Lebesgue and Vitali are fundamentally equivalent; Lebesgue claims that:

Main Theorem – A real bounded function $f(x)$ is Riemann integrable if and only if its set of discontinuities is a null measure set.

It is merit of Lebesgue to have borrowed the definition of a null measure set from Borel: it is a set such that its points can be enclosed in a finite or enumerable number of intervals whose total length is as small as we want. On the other hand stating the same theorem independently Vitali requires that the set of discontinuities has *estensione minima nulla*,⁴ where the minimal extension of a set is the g.l.b. of the sum of the lengths of the intervals of a finite or enumerable system that covers the set without internal points in common. This is equivalent to the definition of Lebesgue's external measure (Lebesgue 1904 a, 104), so the two definitions are completely equivalent.

Both the demonstrations start from the du Bois-Reymond's criterion, where the notion of the *oscillation*⁵ of a real function $f(x)$ in a point y as the difference between the maximum and the minimum limit of $f(x)$ in y is used:

A real bounded function $f(x)$ in the interval (a,b) is integrable if and only if for every $\varepsilon > 0$ the set of the points where the oscillation is greater than ε is an "integrable" (according to Lebesgue-du Bois-Reymond) or a "rinchiudibile" (according to Vitali) set.

² For a biography of Vitali see the introduction by Pepe to (Vitali 1984,1-24) and also (Pepe 1984) and (vaz Ferreira 1991).

³ Tricomi said about him: "fu essenzialmente un *self made man* che, per buona parte della sua carriera, lavorò quasi senza contatti con altri scienziati, e gli capitò di arrivare simultaneamente ad altri, ma indipendentemente da loro, specie dal Lebesgue, a fondamentali risultati di teoria delle funzioni di variabile reale, che altrimenti gli avrebbero data fama mondiale" (Giusti e Pepe 2001, 152).

⁴ Also Vitali in his paper *Sui gruppi di punti* (Vitali 1960, 139), where develops his measure theory independently but equivalently to Lebesgue, quotes (Borel 1898).

⁵ Du Bois-Reymond (1831-1889) introduces the "limits of the indetermination" of a series and their difference since 1870 (Hawkins 2002, 26). In 1875 also Ascoli (1843-1896) gives a punctual definition of oscillation and furnishes the following condition:

A function is integrable in $[a,b]$ if and only if for every $\sigma > 0$ and for every sequence of decompositions of $[a,b]$, G_1, G_2, G_3, \dots in partial intervals such that the maximum width of the intervals of the decomposition G_n tends to 0, the sum of the measures of the intervals containing points where the oscillation is equal or greater than σ tends to 0 when n tends to ∞ . (For the proof see (Ascoli 1875) and (Letta 1994)). The preceding condition is equivalent to the criterion established by du Bois-Reymond in 1882. (Ascoli 1875) is written in Italian and is difficult to read, perhaps this is why it passed unnoticed: Lebesgue and also the Italian Vitali refers always in their works to du Bois-Reymond criterion and do not quote it. It is also worth noticing that in the same year in which Darboux Memoir on discontinuous functions appears, in (Ascoli 1875) the existence of upper and lower integrals of a bounded function is proven.

Both words in the preceding criterion, “*integrable*” or “*rinchiudibile*”, indicate a Peano Jordan null set. But, while, as we will see, Vitali devoted to the subject two papers when he was youth and a third one in the years of maturity, Lebesgue claimed, after remembering that yet in a Note of C.R. in 1901 he demonstrated the necessity of the condition: “*J’attache d’ailleurs assez peu de fût à cette condition qui est identique, au langage employé près, à celles de Riemann et du Bois-Reymond*” (Vitali 1960, letter of February, 16th, 1907 from Lebesgue to Vitali). In the *Leçons*, after the demonstration of du Bois-Reymond’s criterion as a consequence of Riemann criterion, Lebesgue proceeds quickly to prove the necessity: if $f(x)$ is integrable, then for du Bois-Reymond criterion the set

$$G(\varepsilon) = \{x : \text{oscill } f(x) \geq \varepsilon\} \text{ is integrable for } \varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \dots$$

On the other hand the set of discontinuities coincides with $G(1) \cup G(\frac{1}{2}) \cup G(\frac{1}{3}) \cup \dots$ then it has null Lebesgue measure. On the contrary if the set I of discontinuities of $f(x)$ has null measure then for every $\varepsilon > 0$ a sequence of intervals (I_n) exists such that $\cup I_n$ contains I and the sum of the lengths of such intervals is $< \varepsilon$. Consider the set $\{x : \text{oscill } f(x) \geq \varepsilon\}$: it is contained in I and, by a Baire’s result, is closed (Baire 1899, 10), therefore a finite number of the previous intervals cover it and therefore it is *integrable*. So, by the du Bois-Reymond criterion, also the sufficiency of the condition is proved.

The proof by Vitali is fundamentally the same, save for some little variations in the name of the notions at stake. It is interesting to remember that, in the paper *Sulla condizione di integrabilità delle funzioni* (Vitali 1903), written one year before, Vitali furnishes a criterion with a different statement, where only *rinchiudibili* sets are involved:⁶

Criterion 1 - A real and finite function $f(x)$ is integrable in an interval (a,b) if and only if the set of its singularities is such that every closed subset of it is rinchiudibile. 4

The proof is founded basically on the du Bois-Reymond’s criterion and on the following theorem proved by Vitali:

Criterion 2 - If $f(x)$ is a real finite function, integrable in the interval (a,b) , then every real and finite function $\varphi(x)$, continuous in the points where $f(x)$ is continuous, is integrable in (a,b) .

Proof of Criterion 1 - The condition is sufficient because if it holds then the set of the points where the oscillation of $f(x)$ is greater or equal to σ , being a closed set contained in the set of the singularities, is *rinchiudibile* and therefore by du Bois-Reymond’s criterion $f(x)$ is integrable.

Conversely, if $f(x)$ is integrable and D is a closed subset of the set of its discontinuity points, then it is possible to construct a function $g(x)$ with a jump greater or equal to σ in the points of D and continuous elsewhere: such a function is integrable by Criterion 2 (that is not proved here) and therefore by the du Bois-Reymond’s criterion the set D is integrable.

It is interesting to remember that Vitali deals again with this subject in (Vitali 1927), in the years of his maturity, when he furnishes a new demonstration of the main theorem. The proof, we give the only if part of it, is very simple because, even if it requires more advanced notions, it is not linked to du Bois-Reymond’s criterion, but is related directly to the Riemann definition of integrability. On this occasion, indeed, Vitali makes use of the following property: given a real, bounded function in an interval (a,b) , the upper (resp. lower) Darboux

⁶ Vitali uses a particular case of a theorem by Osgood (American Journal of Mathematics, 19), that is: If a closed set is the union of a sequence of *rinchiudibili* sets, it is *rinchiudibile*.

integral, as a function of the upper limit of integration x is an absolutely continuous function whose derivative is $f(x)$ in the points x where f is continuous. Now, if a function is absolutely continuous and his derivative is a. e. zero then the function is constant. Therefore if $f(x)$ is a.e. continuous in the interval (a,b) then the difference between the upper and the lower integrals, an absolutely continuous function, has a.e. null derivative so it is a constant: but this difference is zero if $x=a$ and therefore upper and lower integrals coincide in all of (a,b) , that is the function is integrable.

Vitali-Lusin's Theorem

In 1904-05 great interest was devoted to the study of Baire classes by several authors: in (Lebesgue 1904 b) H. Lebesgue proves again, but this time without the use of transfinite ordinals, the fundamental theorem already established by Baire that a function is of class 1 if and only if it is punctually discontinuous.

In 1905 G.Vitali in a brief paper, *Una proprietà delle funzioni misurabili* proves that:

*Every real measurable function is the sum of a function of class not greater than 2 and of an a.e. zero function*⁷ (Vitali 1905 a).

The proof is based on the observation that every measurable function is the sum of an a.e. zero function and of a function that is the limit of a non-decreasing sequence of first class functions. Such functions can be determined thanks to a theorem nowadays known as Lusin's Theorem, which furnishes a characterization of Lebesgue measurability and is established in this paper by Vitali for the first time. Lusin (1883-1950) in his turn will state and prove it again independently in a special paper in 1911.

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⁷ In (Volterra 1881 a) it is proven that, given a finite number of punctually discontinuous functions (that is first class functions by Baire theorem) defined on the same set D , the points where they are all continuous is a dense subset of D . In (Lebesgue 1904 b) this fact is generalized to the case of a sequence of functions. On the other hand, Young (1863-1942) proves in 1911 that every real bounded measurable function is the sum of a first class function and of an a.e. null function, without quoting the results of Vitali and Lebesgue (Young 1911). Indeed Young proves that every bounded measurable function $f(x)$ is equivalent for purposes of integration to an upper bound of an increasing sequence of upper semicontinuous (u.s.) functions and to a lower bound of a decreasing sequence of lower semicontinuous (l.s.) functions (Young 1911, 16). Now an u.s. or a l.s. function is a first class function and therefore, by Baire theorem, is pointwise continuous. By the preceding theorems by Volterra and Lebesgue, given an increasing sequence of u.s. functions and a decreasing sequence of l.s. functions, there is a dense set E where all the functions are continuous and therefore f being the upper limit of the first sequence is l.s. in E . and being the lower limit of the second sequence is also u.s. in the same set E , then f is punctually discontinuous, that is belongs to the Baire first class.

It is worth noticing that, some years before, the argument was discussed in some letters from Lebesgue to the young Fréchet (Taylor Dugac 1981); in the letter of Dec. 26th, 1904, Lebesgue writes: "You start from a very beautiful observation, that is if f is the limit of f_n and f_n is the limit of f_{np} then it is possible to select some f_{np} in such a way they have f as their limit except for a set of null measure. Then you conclude that if f_n is limit of polynomials (except on a null measure set) also f is limit of polynomials. It follows that every Baire's function can be represented a. e. by a sequence of polynomials".

Since every measurable function a.e. coincides with a Baire function it follows by the previous Fréchet theorem that every real measurable function of real variable is the a.e. limit of a sequence of continuous functions (Taylor Dugac 1981, 150 and following, where two proofs by Lebesgue are given). It is also interesting the remark by Lebesgue (Taylor Dugac 1981, 159, letter of January, 22th, 1905): "It is clear that the foregoing statements implies: 'f equal a.e. to a convergent sequence of polynomials', but that does not mean 'f a.e. the same of a function that is everywhere of the first class'".

It is interesting to remember that Tonelli in 1910 proves that every Lebesgue summable function of two variables is a.e. the sum of a series of polynomials (Tonelli 1910).

Here is the statement of Vitali-Lusin's Theorem:

If $f(x)$ is a measurable function in an interval (a,b) whose length is l , then, for every $\varepsilon > 0$ a closed subset P of (a,b) exists such that the restriction f_P of f to P is a continuous function.

In a footnote Vitali claims that this theorem looks like another theorem at which the following papers hint: Borel, *Un théorème sur les ensembles mesurables*. C. R., 7. 12. 1903, Lebesgue, *Sur une propriété des fonctions*, C.R., 28. 12. 1903 and Lebesgue, *Leçons*, footnote at p.125. Vitali adds that the author, that is Lebesgue, did not give yet explicit demonstration of it.⁸

The proof of Vitali-Lusin's Theorem is based on a proposition stated and proved by Vitali himself:

If $f(x)$ is a measurable function in a subset G , whose measure is M , of a bounded interval (a,b) , then given the numbers $\varepsilon > 0$ and $\sigma > 0$, there exists a perfect subset $G_1 \subseteq G$, whose measure is greater than $M - \varepsilon$ and there exists a segment δ , such that in every interval whose measure is $\leq \delta$, the function, restricted to the points of G_1 , has an oscillation $\leq \sigma$.

Now Vitali proves the main theorem. Indeed let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s, \dots$ be positive numbers such that the sum of the series $\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_s + \dots$ is less than ε . Moreover let $\sigma_1, \sigma_2, \dots, \sigma_s, \dots$ be an infinitesimal sequence of positive numbers. By the preceding proposition there exist a perfect set G_1 such that $m(G_1) > l - \varepsilon_1$ and a number δ_1 such that in every interval whose length is $\leq \delta_1$ the function $f(x)$, restricted to G_1 has an oscillation not greater than σ_1 . Now for the preceding proposition applied to G_1 there exist a perfect set $G_2 \subseteq G_1$, such that $m(G_2) > m(G_1) - \varepsilon_2$, and a number δ_2 such that in every interval whose length is $\leq \delta_2$ the function, restricted to G_2 , has an oscillation $\leq \sigma_2$, and so on.

The intersection of the sets G_1, G_2, \dots is a closed set I , whose measure is $m(I) > l - \sum_{s=1}^{\infty} \varepsilon_s > l - \varepsilon$.

The function f is continuous in the points of I : indeed, for every positive ω there exists $\sigma_s < \omega$ and for every x_0 , in every neighborhood of x_0 whose length is $\leq \delta_s$ the function restricted to G_s and a fortiori to I , has an oscillation $\leq \sigma_s < \omega$.

Now we will expose the proof of Vitali-Lusin's theorem given by Lusin in 1911 in his Note on the *Comptes Rendus* where his results on measurable functions already published in Russian the year before are exposed. A characterization of the class of Baire functions is obtained in a very simple way since it is based on Severini-Egoroff's Theorem (Lusin 1912).

By the way we remember that this theorem was proved before by Severini in 1910 and after independently by the Russian mathematician Egoroff (1869-1931) in 1911: the brief paper by Egoroff starts quoting Borel (Borel 1905, 35) and Riesz⁹ treatment of the relations between a.e. convergence and measure convergence (Egoroff 1911) and the author obtains his simple proof by using a consequence, obtained by Lebesgue, of the implication a.e. convergence \Rightarrow measure convergence.¹⁰

⁸ In (Lebesgue 1904 a, footnote p. 125) the following, statement appears: "Toute fonction mesurable est continue, sauf aux points d'un ensemble de mesure nulle, quand on néglige les ensembles de mesure ε , si petit que soit ε ". Analogous statement one can find in the Note *Sur une propriété des fonctions* (Lebesgue 1903). In it Lebesgue observes that the preceding property, he calls Borel property, since this author has pointed out it in a Note of the C.R. (December 7th, 1903), characterizes the measurable functions.

⁹ F. Riesz, *Sur les suites de fonctions mesurables*, C.R. t.148, p.1303.

¹⁰ An equivalent statement is given for the series, as we will immediately see, in (Lebesgue 1903), for the proof see (Lebesgue 1906, 9) and (Borel 1905,35).

Also Severini's proof is based on Lebesgue's result, but it is inside another demonstration: for this reason and also because it is written in Italian, the theorem is known out of Italy with the only name of Egoroff.¹¹ The statement of Severini-Egoroff's theorem is the following:

If (f_n) is a sequence of measurable functions in $[0,1]$ a.e. converging to a function $f(x)$ then for every $\varepsilon > 0$ there exists a closed set P such that $\text{meas}(P) > 1 - \varepsilon$ and the sequence uniformly converges to f in P .

Here is the proof by Severini for the series, completely independent from the context where it is inserted: consider in an interval (a,b) the a.e. convergent series $\sum f_n(x)$. By a result in (Lebesgue 1903), given two positive numbers ε and σ , it is possible to determine an increasing sequence of natural numbers $n_1, n_2, \dots, n_i, \dots$ such that, if H_{n_i} is the set of the points where some remainder of the series starting from the n_i^{th} is $\geq \frac{\varepsilon}{2^i}$ then $m(H_{n_i}) \leq \frac{\sigma}{2^i}$. Now if i' is a natural number such that $\frac{\sigma}{2^{i'-1}} \leq \tau$ and if $H_{i'} = H_{n_{i'}} \cup H_{n_{i'+1}} \cup \dots \cup H_{n_{i'+h}} \cup \dots$ then in $(a,b) - H_{i'}$ the series uniformly converges.

Indeed for every $g > 0$ if i'' is such that $i'' \geq i'$ and $\frac{\varepsilon}{2^{i''}} \leq g$, then for every $n \geq n_{i''}$ it is $|\sum_{k \geq n} f_k(x)| < \frac{\varepsilon}{2^{i''}} \leq g$ for every $x \in (a,b) - H_{i''} \supseteq (a,b) - H_{i'}$; moreover

$$m(H_{i''}) \leq m(H_{i'}) \leq \frac{\sigma}{2^{i'}} + \frac{\sigma}{2^{i'+1}} + \dots + \frac{\sigma}{2^{i'+h}} + \dots = \frac{\sigma}{2^{i'-1}} \leq \tau.$$

The proof by Egoroff is very similar. It is possible to find a proof of Vitali Lusin's Theorem, in (Lebesgue 1903), where the "propriété de M. Borel" stands for a new type of continuity arising from continuity when passing to the limit: 7

In order to prove that the functions defined at the present time satisfy Borel's property it is enough to remark that the continuous functions satisfy this property and it is preserved when passing to the limit.

And Lebesgue adds, without demonstration:

Every convergent series of measurable functions is uniformly convergent if we omit a set whose measure is ε , ε being as small as we want.

Borel and Lebesgue already enunciated in 1903 not only Vitali Lusin, but also Severini Egoroff Theorem!

The very simple proof by Lusin is similar to the preceding one given by Lebesgue: it is based on the fact that if f is of class 1, by Severini Egoroff theorem, the following property holds:

For every $\varepsilon > 0$ there exists a closed set P such that $\text{meas}(P) > 1 - \varepsilon$ and f is continuous on P .

Now Lusin proves that the class of functions for which this property holds is closed under a.e. convergence ("elle se conserve à la limite", as Lebesgue says). It follows that all Baire's functions verify this property.

¹¹ Only in 1924, in the brief note *Su una proposizione fondamentale dell'analisi*, (B.U.M.I.(2), Vol. 3, p. 103-104) L. Tonelli (1885-1946) credits Severini for the first proof of Severini-Egoroff Theorem.

Moreover, every Lebesgue measurable function coincides a.e. with a Baire's function and Vitali-Lusin's Theorem is proved.

It is interesting remember that some years after, L. Tonelli introduces the definition of *funzione quasi-continua* (almost continuous function) in the following way (Tonelli 1921, 131):

A function f defined in the interval (a,b) is called an almost continuous function if for every $\varepsilon > 0$ there exists a closed subset H of (a,b) such that the restriction of f to H is continuous and $m(S-H) < \varepsilon$.¹²

Many authors were interested in this argument. Sierpinski, in 1916, in order to give an elementary proof of Lusin's Theorem, gives a new definition of measurable set based on set theory. Precisely he defines the sets of null measure as usual, then for every set he calls it measurable if it is the union of a closed set and of a set of null measure: he does not give a definition of measure but he can define in this way measurable functions (Sierpinski 1916).

Absolutely Continuous Functions

In (Lebesgue 1904 a) the author studies with care functions of bounded variation whose main properties he describes since Chap. IV. But only in Chap. VII, after defining the notion of summable function, he displays their more interesting properties. In his thesis he had proved that if a function $f(x)$ is everywhere derivable and his derivative is bounded then the derivative is Lebesgue integrable and the following equality: $f(x)-f(a)=\int_a^b f'(x)dx$ holds, but he had been unable to extend this result to whatever unbounded derived number. This is accomplished in (Lebesgue 1904 a) in the following way:

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Assume a derived number of a function is finite,¹³ then it is summable if and only if the function is of bounded variation. If this is the case its total variation coincides with the integral of the absolute value of the derived number.¹⁴ (Lebesgue 1904 a, 122-123)

Lebesgue claims also that the fundamental relation $f(x)-f(a)=\int_a^b \mu(x)dx$ holds, where μ is the derived number that is supposed to exist. The proof is based upon the notion of chain

¹² Without reference to Vitali-Lusin's theorem, in 1921 Tonelli proves that every almost continuous function is Lebesgue measurable, but the converse does not hold, because: *non sono definite quelle funzioni nelle cui definizioni si fa uso del principio delle infinite scelte arbitrarie.*

Indeed, this author introduces the class of measurable sets in a completely new way in order to avoid any use of choice axiom: obviously it is a more restrict class of sets than the class of Lebesgue measurable sets. For the same reason he considers exclusively analytically expressible functions (as defined by Lebesgue): he proves that they are all almost continuous and defines his integration theory, less generally than Lebesgue, based on the new measure space and quasi-continuous functions. So, his definitions of measurable sets and functions and his integration theory are closer to Borel's than Lebesgue's theory and they agree with the classical contemporary mathematics.

Vitali represents an opposite tendency with respect to choice axiom and uses it in a decisive manner to prove the existence of a not Lebesgue measurable set (Vitali 1905 e).

¹³ Observe that if it possesses only discontinuity of first type it is Riemann integrable. Indeed in 1908 L. Tonelli wrote *Discontinuità di 1ª specie e gruppi di punti*, (Tonelli 1960, 69-74) where he proves that the set of the discontinuities of first type of a real function of a real variable is at most countable. Then if a function has no second type discontinuities, for (Dini 1878, 244), it is Riemann integrable (as Riemann's function).

¹⁴ This last equality is wrong in general if the derivative number is not finite in every point: indeed if $f(x)$ is the Cantor function then its variation is 1, but the integral of its derivative number is zero (see also the statement and the proof in (Levi 1906, 359)).

of intervals he defines at p. 62 and the use of transfinite ordinals. He extends also the fundamental theorem proved by Dini (Dini 1878, 278)¹⁵ for Riemann integrable functions to Lebesgue summable functions, but it is surprising since, while Dini's result agrees with the Vitali-Lebesgue's theorem that a Riemann integrable function is almost everywhere continuous, no similar information is available for summable functions (for example the characteristic function of the rational numbers of the interval $[0,1]$ is summable but totally discontinuous):

If f is summable on $[a,b]$, then the derivative of the integral $F(x) = \int_a^x f(t)dt$ is a. e. equal to $f(x)$.

He concludes his book by adding that:

every bounded variation function $f(x)$ has a. e. a finite derivative that is summable. But in general $f(x)$ is not a primitive of its derivative, as it is possible to see for example with the Cantor function (Lebesgue 1904 a, 128-129).

Then Lebesgue claims, in a footnote and without demonstration that:

In order that a function is an indefinite integral its total variation in every countable number of intervals of total length l has to tend to zero with l .

And also:

If in the statement at page 94 one does not suppose $f(x)$ bounded nor $F(x)$ with bounded derivatives numbers, but only the previous condition, then one obtains a definition of integral equivalent to that developed in this Chapter and applicable to all summable functions bounded or not.

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Now at page 94 there is the following definition whose particular cases are Duhamel's and Riemann's definitions of integral:

A bounded function $f(x)$ is called summable if there exists a function $F(x)$ with bounded derivatives numbers that admits $f(x)$ as its derivative except for a set of values of null measure. The integral in (a,b) is then, by definition, $F(b)-F(a)$.¹⁶

In 1905 Vitali writes the paper *Sulle funzioni integrali* (Vitali 1905 d). Where he focuses on the nomenclature and gives the fundamental definition of *absolutely continuous function* that will be quoted in all the subsequent literature on this topic.

Let $F(x)$ be a real function defined in the interval (a,b) and let (c,d) be an interval contained in (a,b) : Vitali calls *increment of $F(x)$ in the interval (c,d)* the difference $F(d)-F(c)$ and

¹⁵ If $F(x) = \int_a^x f(x)dx$ (Riemann's integral) then the four derivatives of F are all finite and (Riemann) integrable and if g is one of them, then $F(x)-F(a) = \int_a^x g(x)dx$.

¹⁶ The passages from the *Leçons* above can be found also in (Lebesgue 1907 b, 286 Note 1), where Lebesgue claims priority about the extension of the fundamental calculus theorem. In the same paper he says that B. Levi pointed out to him the paper by Vitali where the same theorem is enunciated and proved. Lebesgue quotes many other works by Vitali about Lebesgue's theory and concludes: *These points of contact show that the theorems we are talking about are very natural and they arise to all those are studying such questions.*

calls *increment of $F(x)$ in a set of pairwise disjoint partial intervals* the sum of the increments of $F(x)$ in every interval.

Then he calls *absolutely continuous* a function $F(x)$ if for every $\varepsilon > 0$ there exists $\sigma > 0$ such that the sum of the absolute values of the increments of $F(x)$ in a set of pairwise disjoint intervals such that the sum of their lengths is less than σ , is less than ε .

He calls *integral function* a function $F(x)$ if there exists a (Lebesgue) summable function $f(x)$ in (a,b) such that for every $x \in (a,b)$ it is: $F(x) - F(a) = \int_a^x f(t) dt$.

At this point Vitali shows that:

a function $F(x)$ in (a,b) is an integral function if and only if it is absolutely continuous.

In this way Vitali expresses in an elegant manner the only sketchy thought in the Lebesgue footnote, and furnishes a precise demonstration of it, including the case of not bounded derived functions.

In order to obtain the proof of the fundamental theorem, Vitali establishes some partial results that one can find also in the *Leçons*: for example he proves that if a function is absolutely continuous then it is of bounded variation, (*toute intégrale indéfinie est à variation bornée*, in the *Leçons*) and also that it is not true the converse of the preceding proposition, that is there are continuous functions of bounded variation that are not absolutely continuous like the Cantor function (the $\xi(x)$ function of the *Leçons*): indeed it is continuous and not decreasing, therefore it is a function of bounded variation, but for every $\sigma > 0$ there exists a set of intervals whose union contains the Cantor set such that the sum of all the intervals is less than σ : The increment of the function in it is equal to 1 and therefore the Cantor function is not absolutely continuous.

Borrowing a theorem from the *Leçons* he modifies slightly, Vitali proves that 10

1) every derived number of a continuous function with bounded variation is a. e. finite. Then he proves that

2) if a function $F(x)$ is absolutely continuous and zero is a.e. between the two derived right numbers then $F(x)$ is a constant (Vitali 1905 d, &6).

Moreover, generalizing an important result of the *Leçons* to unbounded functions:

3) the indefinite integral of a summable function has a.e. this function as its derivative (&7).

4) the lower right-derived number A_d of a continuous non-decreasing function $F(x)$ in the closed interval (a,b) is summable (&8).

5) an absolutely continuous non-decreasing function is an integral function (&9).

Finally, since every absolutely continuous function is the difference of two absolutely continuous non-decreasing functions, the theorem is proved.

In 1916 L.Tonelli, (Tonelli 1916a) in the paper *Sulla ricerca delle funzioni primitive*, will give a simpler proof than Vitali's one of the fact that an absolutely continuous function is the integral of its derivative, a proof more elementary also than the first rough one by Lebesgue, based on intervals chains and resorting to the transfinite (Lebesgue 1904, 121-22). Tonelli's proof is obtained approximating the given function of bounded variation, whose graph is therefore rectifiable, by piecewise linear functions.

Exchange of Opinions B. Levi – Lebesgue

For its conceptual characteristic breaking with the traditional mathematical culture, the apparition of the *Leçons* by Lebesgue gave rise to a series of comments. Since Lebesgue's admirable and extensive work often shows some obscure aspects, B. Levi, in 1906, begins an interesting exchange of opinions with the French mathematician, writing four Notes on the *Rend. Acc. dei Lincei*, justifying he himself some enunciates by Lebesgue, and suggesting that the demonstration of some other propositions has to be improved.

Levi intends to submit the more questionable propositions of (Lebesgue 1904 a, 122-23) to a careful analysis, following anyway Lebesgue's methods.

Namely in the first Note he begins considering the proposition by Lebesgue:

The integral of one of the derived numbers of a function (supposed finite) does exist if and only if the function is of bounded variation; if this is the case the indefinite integral of that derivative number is a primitive.

Now, Levi says, in this proposition the hypothesis that the derivative number is bounded is essential and therefore the proof given by Lebesgue is faulty. In order to give a more careful proof of it Levi proves some auxiliary facts:

The lower or upper, right or left derived numbers of a continuous function belong to the Baire second class; therefore, they are measurable (Levi 1906 b, 433).¹⁷

If $f(x)$ is a continuous function and in every point the right and left derivatives exist then the set of the points where they are different is enumerable (Levi 1906 b, 437).

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Then he proves again the proposition of (Lebesgue 1904 a, 120), that is:

The indefinite integral of a bounded measurable function $f(x)$ has a.e. a derivative equal to $f(x)$.

He demonstrates with care that the previous theorem holds also in the case $f(x)$ summable but not bounded (pp. 674-681).

In all the Notes he continues his strong analysis of Lebesgue's work, studying in detail the relations between the behavior of derivatives and their primitive functions. And particularly he corrects the previous Lebesgue's statement in the following way:

a function of bounded variation is a primitive of one of its derived numbers if this number is bounded (and not merely finite).

This is analogous to some other propositions in (Lebesgue 1904 a, 123-125) that are rather ambiguous for the presence of the word "finite" in place of "bounded". One of the main results of Levi's research about this topic is the following very interesting characterization of the integral functions:

Let $f(x)$ be a continuous function in the interval $[a, b]$ and let $u(x)$ be one of its derivative functions. Then the fundamental theorem of the calculus holds if and only if the set of the

¹⁷ In (Baire 1899, 64) it is proved that if a function is derivable then its derivative is in the first class.

values assumed by $f(x)$ in a zero measure set has zero measure and $u(x)$ is a summable function (Levi 1906 b, 358).¹⁸

The fundamental theorem by Levi in the case the derived number is not bounded is given in (Levi 1906, 359), using the notion of *reducible* set that is a set such that one of its derived sets is empty:

Let $u(x)$ be one of the derived functions of the continuous function $f(x)$ in the interval $[a,b]$: if the set of the points where $u(x)$ is infinite is at most reducible, then the necessary and sufficient condition in order that $u(x)$ is Lebesgue summable, is that $f(x)$ is of bounded variation. Then $f(x) = \int_a^x u(t)dt + C$, where C is a constant.

Observe that by the characterization of the integral functions given by Vitali, we can obtain by this theorem the following interesting proposition:

a continuous function is absolutely continuous if one of its derived functions is summable and the points where it is infinite is at most reducible.

After the long proof, Levi observes that he has preferred a way of proof different by that used by Lebesgue: the French mathematician in his proof, in order to calculate the total variation of the given function, considered a sum extended to an enumerable chain of intervals. Remember that a chain of intervals, defined in (Lebesgue 1904 a, 63), is an ordered set of disjoint intervals of the real line. Every interval is contiguous to the preceding one if it exists; otherwise it has as first extreme the upper bound of the second extremes of the preceding intervals. The operation of extending a sum to an enumerable chain of intervals seems to Levi an elegant and suggestive but always very delicate operation, resorting to the transfinite, about which Levi has some reservations.

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Lebesgue, quite irritated, immediately answers, (Lebesgue 1907 a, b), meeting some Levi's objections and adding some intermediate reasoning, but also explaining some definitions he himself used.

He claims that, while a bounded derivative of a function is not necessarily Riemann integrable, as Volterra proved with a counter example, on the contrary:

the indefinite Lebesgue integral of a bounded derivative always exists and is one of its primitive functions.

If the derivative is not bounded he remembers that in the *Leçons* he gave the following propositions:

The indefinite integral of an everywhere finite (bounded or not) derived number is one of its primitive functions if and only if the derived number is Lebesgue summable.

Moreover:

¹⁸ Observe that, by Vitali's theorem about integral functions, this proposition implies the following Banach-Zarecki Theorem:

If $f(x)$ is continuous, of bounded variation and the Lusin's condition (N) holds (that is for every subset E of $[a,b]$ $m(E)=0 \rightarrow m(f(E))=0$) then $f(x)$ is absolutely continuous on $[a,b]$.

It is clear that Lusin's (N) condition is already present in the proposition by B. Levi.

The indefinite integral of an everywhere finite derived number is one of its primitive functions if and only if the primitive functions are of bounded variation.

In this way Lebesgue corrects the analogous proposition of the *Leçons* simply by removing the part regarding the representation of the function having the derived number by its integral.

Lebesgue adds that, in general, both the derived number is finite or infinite, the problem of the determination of a primitive function of it does not make sense. Indeed, Lebesgue claims, Hans Hahn has proved that there exist two functions, f and g , having everywhere the same derivative but $f-g$ is not a constant; if the derivative is not finite it is impossible to determine the function up to a constant.

It is worth noticing that Lebesgue assures that the previous propositions are absolutely correct and they were given a proof also by B. Levi (Lebesgue 1907 b, 284 Note 1).

But he does not agree with the drastic opinion by Levi about the use of transfinite chains of intervals in his demonstrations. This is a crucial point in the polemic. Lebesgue refutes his opponent's argument defending his procedure. In particular Levi has observed that if the extremities of the intervals constitute an irreducible set, as in an example he exhibits and as in the Cantor set, the results are no more reliable. Lebesgue answers simply that he was already acquainted with that.

We now finish the exposition of the exchange of ideas between Lebesgue and B. Levi, by reporting a summary of the subsequent development of the problem of the research of the primitives that is possible to read in a paper of 1920 by L. Tonelli: *Sulla ricerca delle funzioni primitive*, (Tonelli 1960, Vol. I, 360-376). In it the Author deals with the problem in the most general setting, making a survey of the situation. He recapitulates that if a function has a bounded derivative then this derivative is Lebesgue (not always Riemann) integrable and the equality: $f(x)-f(a)=\int_a^b f'(x)dx$ holds; moreover, Vitali showed that this equality holds for every absolutely continuous function.

But there are some not absolutely continuous functions that possess a finite derived-number or even a finite derivative at every point, which is not Lebesgue integrable.¹⁹ The problem to determine a primitive function in this case was faced by Denjoy and solved by a new method of integration he called *totalisation*.

It was not possible to say that the research was finished at this point since functions exist which have derivative or a given derived number that is not always finite. Now, while in the preceding cases the solution is proved to be unique, in the last case the problem may be undetermined. If for example a function has an upper right-derived number equal to $+\infty$ ($-\infty$) at every point of a perfect set P , if we add to it a function $\varphi(x)$ that is constant in all the intervals contiguous to P and increasing (decreasing) to the right at every point of P that is not the first extremum of a contiguous interval, the sum $f(x)+\varphi(x)$ and $f(x)$ have always equal upper right-derived numbers, but $f(x)$ and $f(x)+\varphi(x)$ do not differ from one another only by a constant. Examples of such functions (mentioned by Lebesgue) were given by H. Hahn, B. Levi and others.

Remember that B. Levi gives the following sufficient condition to the point:

Two continuous functions $F(x)$ and $G(x)$ defined in the same interval differ at most by a constant if they have equal derivative in every point of the interval, the derivative being finite everywhere except at the points of a set E such that $m[(F-G)(E)]=0$ (Levi 1906, 558).

¹⁹ For example the function defined in $[0,1]$ in the following way: $f(0)=0$, $f(x)=x^2 \sin \frac{1}{x^2}$ for $x \neq 0$, has a not summable derivative that is finite in every point.

As a counterexample let $F(x) = \xi(x)$ and $G(x) = 2\xi(x)$ where $\xi(x)$ is the Cantor function. $F(x)$ and $G(x)$ are continuous and their derivatives coincide everywhere; they are infinite in the points of the Cantor set C , but $m[(F-G)(C)] = 1$. Indeed $G(x) - F(x)$ is not a constant. Observe that C is perfect.

Well, Tonelli, by using the Denjoy integration method, proves that:

A necessary and sufficient condition in order to a continuous function be determined, up to a constant, by a derived number is that:

- 1) *this number exists a.e.;*
- 2) *the set of points where this number is infinite does not contain a perfect set.*

Term by Term Integration of Equi-Absolutely Continuous Sequences of Integrals

Only in 1907 Lebesgue sends a letter²⁰ to Vitali about the papers of the Italian mathematician on measure and integration theory; indeed, he complains: “I have always had knowledge of your papers (when I have had) by chance and after their publication”. He notices that there are very many points in common in their respective papers: with regard to absolute continuity and the relevant theorem, Lebesgue underlines his priority, but acknowledges Vitali as who threw light upon the result he himself had mentioned on the contrary only in passing; but above all he praises Vitali since he has transposed the definition of absolute continuity to a useful condition to integrate series term by term.

Indeed, in February 1907 Vitali published the paper *Sull'integrazione per serie* (Vitali 1907), a very famous and quoted paper, where the definition of absolute continuity is extended to a family of functions in the following way: given a family of functions, their integrals are called *equi-absolutely continuous* if

for every $\varepsilon > 0$ there exists $\sigma > 0$ such that the absolute value of the integral of whatever function of the family, extended to whatever measurable set whose measure is less than σ , is less than ε .

After this definition Vitali obtains the following criterion:

A series of summable functions defined in a measurable set of finite measure is term by term integrable if and only if it converges and the integrals of its partial sums are equi-absolutely continuous.

In particular if we have a uniformly convergent series of continuous functions defined in a set of finite measure, then, as was exclusively applied until then, the term by term integration is allowed by the criterion. Lebesgue had yet generalized this result to bounded measurable functions in the *Leçons*, by the following theorem:

If a series converges and its partial sums are uniformly bounded, then it is possible to integrate it term by term.

This theorem is quoted by Vitali, since it is a simple consequence of his general criterion; but Vitali does not notice that by his criterion the more general condition easily follows:

²⁰ Letter of February 16th, 1907 (Vitali 1984, 457).

If a series converges and all its partial sums are less or equal to a summable function, then it is possible to integrate it term by term.

Such a theorem will be enunciated and proved in (Lebesgue 1908). With regard to the same theorem, in a footnote of a subsequent paper (Lebesgue 1909), the French mathematician observes that it is implied by Vitali's criterion; he also remembers that Poincaré had given in 1894-95 some lessons at the Sorbonne about the potential theory, edited with the name of *Leçons sur la Théorie du potentiel newtonien* by Le Roy and Vincent and published in 1899, where a theorem in order to pass to the limit under integral sign is enunciated; in it the hypothesis of a summable majorant plays a crucial role.²¹

It is worth noticing that by Vitali's general criterion, as Vitali himself underlines, also the following Beppo Levi's Theorem (Levi 1906 a), published one year before, follows:

Let $(f_n(x))$ be an increasing sequence of nonnegative measurable functions and let $f(x)$ be its limit. Then

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx.$$

Thus, the determination of Vitali's criterion, together with the more available Lebesgue's and B. Levi's sufficient conditions it implies, concludes the research about the topic.

The Problem of the Reduction of Double Integrals

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Reducing double integrals was a complex problem from the very beginning; the difficulty was linked to the fact that if a function is integrable, both in Riemann and in Lebesgue sense, the partial functions obtained by restricting it to the segments of equation $x=\text{constant}$ and $y=\text{constant}$ are not measurable in general. For integrals in the sense of Riemann among the mathematicians which were interested in the problem there was also Arzelà who gave his contribution to its solution in 1891. For the Lebesgue's double integrals in his thesis Lebesgue had encountered the same difficulty for his integral but he had observed that if the function is Borel measurable in the interval $D=[a,b] \times [c,d]$ then the partial functions $x \rightarrow f(x,y)$ and $y \rightarrow f(x,y)$ are all Borel measurable and if $f(x,y)$ is bounded then for the double integral extended to D we have:

$$\iint f(x,y)dx dy = \int_a^b dx \int_c^d f(x,y)dy = \int_c^d dy \int_a^b f(x,y)dx \quad (*)$$

²¹ Indeed, we can read (Lebesgue 1909, 121):

“Soit l'intégrale $\iint f(x,y,z)dx dy$ étendue à un certain domain S . Supposons:

1° que le contour C qui limite S ne dépende pas de z ;

2° que l'on ait en tout point de S $|f(x,y,z)| \leq \varphi(x,y)$, φ étant une fonction positive;

3° que l'intégrale $\iint \varphi(x,y)dx dy$ étendue au domaine S , ait un sens;

4° enfin, que l'on ait $\lim_{z \rightarrow 0} f(x,y,z) = f(x,y,0)$ quels que soient x et y , pourvue qu'ils restent fixes.

Dans ces conditions, on a la relation: $\lim_{z \rightarrow 0} \iint f(x,y,z)dx dy = \iint f(x,y,0)dx dy$.”

But Lebesgue notices that since there is no demonstration, it is difficult to have an overall view of the situation; some explanations before the enunciate let us believe that Poincaré referred to continuous functions and that $f(x,y,z)$ was supposed to converge uniformly to $f(x,y,0)$ except for some exceptional points. While until then uniform convergence had played a central role when passing to the limit under the integral sign, in this way Poincaré made use of a summable majorant for the first time, allowing the possibility $meas(S)$ infinite.



In the case the integrals in (*) are Riemann's integrals, some mathematicians had showed that there are many exceptions to its validity; in 1907 E. W. Hobson observed that all the counterexamples fail if we consider in (*) the Lebesgue's integral. Consider for example Thomae's example: $D=[0,1] \times [0,1]$, $f(x,y)=1$ if x is rational, $f(x,y)=2y$ if x is irrational: the integrals $\iint f(x,y) dx dy$ and $\int_0^1 dy \int_0^1 f(x,y) dx$ do not exist in Riemann's sense but Hobson pointed out that both exist in Lebesgue's sense and have value 1 (Hawkins 2002, 159).

Hobson did not realize that equality (*) holds in general for a function of two variables if it is integrable in the sense of Lebesgue. Instead this fact was noticed first in 1906 by B. Levi, who in a footnote of a paper about Dirichlet's Principle, claimed, in passing and without proof, starting from some papers of Pringsheim about superficial Riemann's integrals, that if a function f is summable in the unitary square then it is possible to deduce that the linear integrals do exist on every segment of equation $x=\text{constant}$ or $y=\text{constant}$ except at most for a set of lines of zero measure and therefore negligible. Then the double integral can be obtained by means of two successive integrations.²²

In (Fubini 1907) the idea of B. Levi is dealt with apart, and we can find finally the proof of the following most general version of the reduction theorem:

Let $f(x,y)$ a function of two variables, summable in a rectangular set $D=[a,b] \times [c,d]$: then for a.e. $y \in [c, d]$ the partial function $x \in [a, b] \rightarrow f(x,y)$ and for a. e. $x \in [a, b]$ the function $y \in [c, d] \rightarrow f(x,y)$ are summable and () holds.*

Fubini proves that if $f(x,y)$ is summable in D then there exists a Borel measurable function $g(x,y)$ that coincides a. e. with $f(x,y)$ in D . It follows that $f(x,y)$ is linearly summable that is for a.e. $y \in [c,d]$ the partial function $x \in [a, b] \rightarrow f(x,y)$ and for a. e. $x \in [a, b]$ the partial function $y \in [c, d] \rightarrow f(x,y)$ are measurable: Fubini proves that they are also summable and (*) holds. 16

Two years after Tonelli inverts the preceding proposition if the function is nonnegative.

If $f(x,y)$ is a measurable nonnegative function in the set $D=[a,b] \times [c,d]$ such that one of the iterated integrals in () is finite, then $f(x,y)$ is summable and (*) holds (Tonelli 1960, Vol. I, 157).*

It is well known that great difficulties Lebesgue's integration theory encountered to be considered at an international level as a significant mathematical theory (Vitali 1984, 9-10). By means of Fubini-Tonelli theorem and the theorems about the passage to the limit under the integral sign Lebesgue's theory was complete and ready for the applications, so these theorems decreed its triumph.

Contribution by Tonelli to the Problems of the Rectification of the Curves

Jordan in the *C.R. de l'Academie de France* in 1881 and in the second ed. , Vol. 1 of his *Cours d'Analyse* had defined the functions of bounded variation and had proved that a curve is rectifiable if and only the functions which represent it are of this kind.

In 1908, one year after he graduated with first class honors at the University of Bologna discussing a thesis about the polynomials of approximation of Tchebychev, Tonelli²³ writes the paper *Sulla rettificazione delle curve* (Tonelli 1960, Vol.I, 52), where he deals again with the problem of the determination of the length of the curves, this time in the light of

²² B. Levi, *Sul principio di Dirichlet*, Rend. Circ. Mat. Palermo, t.22, 30.

²³ Further information about the life and the works of Tonelli can be found in "Sulla vita e sulle opere di Leonida Tonelli" by Silvio Cinquini in (Tonelli 1960, Vol. I, 1-35).

the new theory of Lebesgue's integral and of the definition by Vitali of an absolutely continuous function.²⁴ He improves the result given by Jordan, establishing the following characterization:

The length L of the curve of equations $x=x(t), y=y(t), z=z(t)$ ($a \leq t \leq b$) is given by:

$$L = \int \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$$

(where the integral is the Lebesgue integral) if and only if the functions $x(t), y(t), z(t)$ are absolutely continuous.

In 1916 he comes back to the subject in the paper *Sul differenziale dell'arco di curva*, (Tonelli 1916 b) where he points out, among other things, that the graph of the Cantor function f is an example of rectifiable curve whose components $x(t)=t, y(t)=f(t), t \in [0,1]$ are of bounded variation but not absolutely continuous. In this case the length is strictly greater than the corresponding integral, and is equal to it plus the total variation of f in the set of the points where the derivative does not exist finite.

Functions of More Variables and Additive Set Functions

In 1915 and after in 1916 de la Vallée Poussin gives a general definitive treatment of additive set functions, their derivatives in a point and the generalization of the fundamental calculus theorem. As we have seen, in the case of one real variable, Vitali had already completely solved the problem in 1905, introducing in the paper *Sulle funzioni integrali* the concept of a real absolutely continuous function of a real variable and characterizing by it the integral functions.

Interesting thing, in 1907, in the paper *Sull'integrazione per serie*, Vitali comes back to the preceding definition, looking at the integral function of a summable function no more as a point function but as an absolutely continuous set function. 17

In the paper *Sui gruppi di punti e sulle funzioni di variabili reali* (Vitali 1908) the Author tries to generalize the fundamental definitions appearing in (Vitali 1905 d) to functions of more variables.

Namely, let $F(x,y)$ be a real function of two variables defined in a rectangle with edges parallel to the coordinate axes. Vitali calls increment of F in the rectangle whose vertices are $(a,b), (x_1,x_2), (a,x_2), (x_1,b)$, the quantity:

$$F(a,b) + F(x_1,x_2) - F(x_1,b) - F(a,x_2);$$

F is called *absolutely continuous in the rectangle R* if for every number $\sigma > 0$ there exists a number $\mu > 0$ such that for every finite set of rectangles contained in R with edges parallel to the coordinate axes the sum of whose measures is less than μ , the sum of the moduli of the increments of $F(x,y)$ in such rectangles is less than σ .

$F(x,y)$ is an *integral function* if and only if there exists a summable function $f(x,y)$ such that:

$$F(x_1,x_2) = \int_0^{x_1} \int_0^{x_2} f(x,y) dx dy.$$

Vitali says that a function F is of *bounded variation* if in whatever manner the rectangle R is divided in a finite number of rectangles with edges parallel to the coordinate axes, the

²⁴ It is interesting remember that in a postcard dated 25th April 1908 Arzelà writes to Vitali "Ho una nota del mio allievo Tonelli relativa alla lunghezza, e che si riattacca all'ultima sua nota. Io sono occupatissimo, potrebbe leggerla lei? è breve" (Vitali 1984, 470).

sum of the moduli of the increments of $F(x_1, x_2)$ in such rectangles is less than a fixed number.

The incremental ratio is defined in the following way:

$$\frac{F(x_1+h, x_2+k)+F(x_1, x_2)-F(x_1+h, x_2)-F(x_1, x_2+k)}{hk}$$

Vitali makes h and k tend to 0 with $h=k$ and calls *numeri derivati destri (sinistri)* in the point P the indetermination numbers of such an expression for h and k tending to 0 and positive (negative). If these limits are finite and equal, their common value is, by definition, the *derivative*.

On the basis of these elements, Vitali claims that it is possible to extend to the case of more variables all the results previously obtained for the functions of one variable.

In order to give an example of a total derivative in the sense of Vitali for a function of more variables, in (Tonelli 1910) the problem of the derivation of a multiple integral is faced in the same order of ideas. Tonelli indeed considers a Lebesgue integrable function of two variables $f(x_1, x_2)$ in a domain A and defines the *first derivative numbers* of the integral of $f(x_1, x_2)$ in the point (x_1, x_2) interior to A as the indetermination limits for $u \rightarrow 0^+$ of the integral:

$$\frac{1}{u^2} \int_0^u \int_0^u f(x_1+u_1, x_2+u_2) du_1 du_2$$

If these limits coincide their common value is called *first derivative* of the integral of $f(x_1, x_2)$ in the point (x_1, x_2) ; if $u \rightarrow 0^-$ in one or in both the integration limits and if there is no indetermination, then it is possible to obtain the *second, third and fourth derivative*. If these four derivatives coincide their common value is the *derivative*. Tonelli is able at this point to prove a generalization to functions of more variable of the analogous result proved by Lebesgue for functions of only one variable, that is: *a Lebesgue summable function $f(x_1, x_2)$ coincides a.e. with the derivative of its integral in the interior points (x_1, x_2) of A . In particular it is the derivative of its integral in the points where it is continuous.*

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In (Lebesgue 1910) the preceding fundamental definitions are generalized to the case of functions of more variables, and Lebesgue extends the results of the last chapter of the *Leçons*, in particular with respect to the multiple integrals derivation.

Lebesgue quotes the young Tonelli's work and, in his paper, makes constantly reference to (Vitali 1908): he inspects Vitali's definitions, visits their positive sides and organizes the new theory; as first thing he defines the notion of *additive set function* taking as a prototype the integral function.²⁵

First an interval function is defined (coinciding with the increment of the given function as in Vitali); indeed in the two-dimensional space Lebesgue considers the interval $I=(A, B)$, where $A=(x_1, x_2)$, $B=(x_1+h, x_2+k)$ ($h>0, k>0$) and defines the function

$$\Psi(I) = f(x_1 + h, x_2 + k) + f(x_1, x_2) - f(x_1 + h, x_2) - f(x_1, x_2 + k).$$

Let f be of bounded variation: then if the interval I is the sum of a finite number of pairwise disjoint intervals I_p then $\Psi(I) = \sum \Psi(I_p)$, that is Ψ is additive. Lebesgue does not highlight the intermediate further steps of the extension of Ψ to a general additive set function, but they are well underlined in (de la Vallée Poussin 1915).

After Lebesgue makes clear the meaning of the derivation in this frame: to this aim firstly he introduces the notion of a *regular family* of domains: a family of measurable

²⁵ "After all, giving an additive and absolutely continuous set function, or giving a bounded and absolutely continuous function of k real variables are two equivalent operations" (Lebesgue 1910, 386-387).

domains is called *regular* if there exists $\mu > 0$ such that, for every set D of the family, it is $m(D) > \mu m(S)$ where S is the smallest hypersphere containing it. By this definition a variable set of the family is not allowed to have an infinitesimal measure if all its dimensions are not infinitesimal.

Then, given an absolutely continuous additive set function, F , Lebesgue defines its derivatives as the indetermination limits of the ratio $\frac{F(E)}{m(E)}$, where m is the Lebesgue measure of E , E varies in the family and the diameter of E tends to 0.

The whole central part of his paper is a re-examination of the definitions and the proofs of Vitali's widely quoted paper of 1908, and a suitable transformation of them leading to the generalization of the theorem of the Italian. Lebesgue says: "J'ai imité son exemple en me bornant tout d'abord aux fonctions F absolument continues".

He then proves the following theorem, analogous to Vitali's theorem:

An absolutely continuous additive set function has a finite derivative a.e., which is summable, and it is the indefinite integral of such a derivative (Lebesgue 1910).

In order to prove it Lebesgue uses a geometric proposition by Vitali,²⁶ the following cover theorem:

Let E be a measurable set: let us suppose that every point of E belongs to an infinite number of arbitrarily small intervals from a family F of intervals. Then it is possible to extract from F a finite or enumerable set of intervals, pairwise without interior points in common, such that the sum of their measures is not less than the measure of E .

In 1908, Vitali notices that an analogous proposition can be proved in the plane, ¹⁹ considering, in place of intervals, a family of squares, or in the space, a family of cubes. Lebesgue proves it for every regular family of domains, but he shows that it is not true for whatever family of domains (Vitali 1908).

Vitali cover theorem is used by Lebesgue in particular to prove that:

If an absolutely continuous additive set function Φ is positive in a quadrable domain D ,²⁷ then there exists in D a set of positive measure where the derivative numbers of Φ are positive.

Indeed if this was not the case, with every point P of D , except the points of a null set, it would be possible to associate a sequence of regular domains δ , interior to D and containing P , whose diameters tend to 0 and such that, for every δ , $\Phi(\delta)/m(\delta) < \Phi(D)/m(D)$. Now by Vitali's theorem there exists a sequence of regular δ , pairwise without interior points in common, covering all of D , except a null set E . Then D can be viewed as the union of the δ and of E and therefore $m(D) = \sum m(\delta)$ and $\Phi(D) = \sum \Phi(\delta)$. It follows:

$$\sum \Phi(\delta) < \sum m(\delta) \Phi(D)/m(D) = \Phi(D),$$

an absurdity.

²⁶ In (Banach 1924) there is a simple demonstration of this fundamental theorem.

²⁷ That is a Lebesgue measurable domain whose boundary is a null set (therefore a quadrable domain is nothing but a Peano Jordan measurable domain).

An important consequence of the preceding proposition which plays a central position in the proof of the main theorem is the following:

Let an absolutely continuous additive set function be such that one of its derivative numbers is a. e. not negative (resp. not positive), then it is constantly not negative (resp. not positive). And therefore if a derivative number is a.e. equal to zero then the function is a.e. equal to zero.

To prove the fundamental theorem Lebesgue considers an absolutely continuous additive set function Φ that is ≥ 0 (this hypothesis is not restrictive since every absolutely continuous function is the difference of two nonnegative). He considers one of the two derivative numbers of Φ , $D(\Phi)$, and determines a lower approximation of the integral function of $D(\Phi)$ in the following way.

Given $\varepsilon > 0$, let $E_i = \{x: i\varepsilon \leq D(\Phi) < (i+1)\varepsilon\}$, $i=1,2, \dots, n-1$, $E_\infty = \{x: D(\Phi) = \infty\}$.

These sets are measurable since Lebesgue has previously proved that:

The derived numbers of an additive absolutely continuous function are measurable functions.

Now, consider a domain F including all the previous ones: let, for every i , ξ_i be a closed set, $\xi_i \subseteq E_i$, such that $m(\xi_i) > m(E_i) - \frac{\varepsilon}{2^i}$ and ξ_∞ a closed set, $\xi_\infty \subseteq E_\infty$, such that $m(\xi_\infty) > m(E_\infty) - \varepsilon$. Let

$$\begin{aligned} f_i(P) &= 0 \text{ if } P \notin \xi_i, & f_i(P) &= i\varepsilon \text{ if } P \in \xi_i; \\ f_\infty(P) &= 0 \text{ if } P \notin \xi_\infty, & f_\infty(P) &= n\varepsilon \text{ if } P \in \xi_\infty. \end{aligned}$$

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Also let
$$\varphi(E) = \sum_{i=0}^{n-1} \int_E f_i(P) dP + \int_E f_\infty(P) dP.$$

It follows that the derivative numbers of φ are less or equal to $D(\Phi)$.

Let $\Lambda(\varphi)$ be the upper derivative number of φ : then it is $D(\Phi) - \Lambda(\varphi) \geq 0$ and therefore, by the preceding proposition, for every n : $\Phi - \varphi \geq 0$. On the other hand, it follows by definition that:

$$\varphi(F) \geq \int_{\xi_1 \cup \dots \cup \xi_n} [D(\Phi) - \varepsilon] dP + n\varepsilon m(\xi_\infty);$$

This relation holds for every n and for every $\varepsilon > 0$. It follows that $m(\xi_\infty) = 0$ and therefore also $m(E_\infty) = 0$. It follows that $D(\Phi)$ is a.e. finite. Let $\varepsilon \rightarrow 0$ and $n \rightarrow \infty$. Since

$$m(F) \geq m(\xi_1 \cup \dots \cup \xi_{n-1}) > m(E_1) + \dots + m(E_{n-1}) - \varepsilon,$$

it is $m(\xi_1 \cup \dots \cup \xi_{n-1}) \rightarrow m(F)$. Therefore we obtain: $\varphi(F) \geq \int_F [D(\Phi)] dP$.

Now Lebesgue puts $D'(P) = D(\Phi)(P)$ if $D(\Phi)(P)$ is finite, and $D'(P) = 0$ otherwise.

It follows $\Phi(F) \geq \varphi(F) \geq \int_F D'(P) dP$ and also $\Phi(E) \geq \varphi(E) \geq \int_E D'(P) dP$ for every domain E . By a similar procedure Lebesgue proves also the converse inequality obtaining $\Phi(E) = \int_E D'(P) dP$ for every domain E . Since this relation holds also for the other derivative number, the two derivative numbers are equal, that is Φ has finite derivative a. e. and coincides with the indefinite integral of such a derivative.

By the fundamental theorem, the integral, that is the function $F: E \rightarrow \int_E f(Q) dQ$, is an additive absolutely continuous function. Lebesgue considers in particular a function f of two

real variables. As usual in order to prove that the derivative of F in P is equal to $f(P)$ it is enough to prove that it is $\lim_{m(E) \rightarrow 0} \frac{\int_E |f(Q) - f(P)| dQ}{m(E)} = 0$ where E varies in a regular family (Lebesgue 1910, 404). As he has proved for the functions of one variable and Tonelli for a function of two variables, now Lebesgue proves that the set of the points P such that this derivative is not $f(P)$ has null measure. Therefore

if $f(Q)$ is a summable function then the derivative of the function $E \rightarrow \int_E f(Q) dQ$ is a.e. equal to $f(P)$.

Lebesgue defines also the *mean density of a measurable set E in an interval (a,b)* as the ratio $\frac{m[E \cap (a,b)]}{b-a}$; and the *density of a set E in a point a on the right* as the $\lim_{b \rightarrow a} \frac{m[E \cap (a,b)]}{b-a}$ if it exists; in an analogous way the density on the left is defined; if the two preceding values coincide they are the *density of the set E in a .*

Lebesgue proves that the density of a measurable set E is a.e. equal to 1 in the points of E and 0 in the complement and therefore coincides a.e. with the characteristic function of E , χ_E .

By the equality $\varphi(a,b) = \int_a^b \chi_E(x) dx = m[E \cap (a,b)]$ it follows that χ_E is a. e. the derivative of its indefinite integral.

Then, Lebesgue deduces in general:

Every summable function of a real variable is a.e. the derivative of its indefinite integral (Lebesgue 1910, 408).

In 1915 de la Vallée Poussin considers a particular family of regular sets and defines the notion of *reseau*: a reseau consists in a sequence of nets by which the space is divided: for example in the case of two dimensions it is possible to obtain a net by two systems of straight lines parallel to the coordinate axes and equally spaced. The subsequent net is obtained adding other two parallel lines equally spaced between two consecutive ones of the preceding net, the distance of the next parallel lines being always in the same ratio with the distance of the preceding lines. 21

The definition of derivative is given in the following way: given the absolutely continuous additive set function, f , de la Vallée Poussin determines for almost all the points P , in a uniform way, a sequence E_n of elements of the reseau such that the point P belongs to every one of them and $\lim_{n \rightarrow \infty} m(E_n) = 0$, then he calculates the lower and upper limits of the ratio $f(E_n)/m(E_n)$. In the case such limits are finite and coincide they are the *derivative of f in the point P .*

Obviously the just now defined derivative depends on the fixed reseau.²⁸

Well, by using such a uniform procedure, de la Vallée Poussin proves the fundamental theorem without using the Vitali covering theorem.

The Quadrature of the Surfaces

Another field where Tonelli gave a great contribution was about the quadrature of the surfaces: in the past the area of a surface S had been often considered as the limit of the areas of some polyhedral surfaces inscribed in S and having S as their limit: but in order that this definition is well given it is necessary to prove that whatever sequence of polyhedrons is considered the corresponding limit is the same. Schwarz and Peano in 1882 gave examples

²⁸ Obviously, this happens in general also for the regular families, but de la Vallée Poussin proves that the choice of the family is influential only in a set of points of null measure (de la Vallée Poussin 1915).

to prove that the previous limit does not exist in general, varying with the particular sequence considered. Subsequently Peano gave, in 1890, in a paper on *Rendiconti dell'Accademia dei Lincei*, the following general definition:

The area of a convex surface is the l.u.b. of the areas of the polyhedral convex surfaces inscribed and the g.l.b. of the circumscribed ones.

This definition is not depending on the Cartesian coordinates, but it is too general and is not useful for applications. Hermite, in his *Leçons*, in 1882, proved, in the way we study nowadays, the formula for the quadrature of a surface of equation $z=f(x,y)$ with (x,y) belonging to a bounded set D : he supposed that the surface had a tangent plane in every point and that this plane varied with continuity. In (Lebesgue 1902, Chapt. IV) the previous definition is extended to a Lebesgue summable function $f(x,y)$ with summable derivatives.

In order to give for the surfaces a definition similar to that of the rectifiable curves it was necessary generalize the definition of function of bounded variation in the case of functions of two variables; this had become a very hard problem since in the case of more variables the definitions by Vitali and Lebesgue of a function of bounded variation turned out to be inadequate.

It is remarkable that some years before Vitali gave his definition of function of two variables of bounded variation Arzelà (1847-1912) made an attempt to define this concept, as follows (Arzelà 1904-05 and 1906-07).

Consider a bounded function $f(x,y)$ defined in a rectangle $ABCD$ whose edges are parallel to the axes and let L be a curve that has A as an extremity and is such that along it x and y are both increasing, at least one of them strictly. Let M be the other extremity belonging to the rectangle and let $M_1, M_2, \dots, M_p=M$ be points of L disposed in the increasing order of x and y . Consider the sum

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$$f(M)-f(A) = f(M_1)-f(A) + f(M_2)-f(M_1) + \dots + f(M_p)-f(M_{p-1}) = P_L - N_L$$

where P_L is the sum of the positive differences and N_L the sum of the absolute values of the negative differences. The Author says that the function $f(x,y)$ is of bounded variation if one of these two sums is bounded when L varies in the class of the prescribed curves and M varies in the rectangle. If this is the case, he proves that also the other sum is bounded and $f(M)-f(A) = P(M) - N(M)$, where $P(M)$ is the upper bound of P_L and $N(M)$ is the upper bound of N_L . Then he proves that the functions $P(M)$ e $N(M)$ are not decreasing if M proceeds along a line of the prescribed ones in the positive direction of x and y . So it is easy to prove that:

the function $f(M)$ is of bounded variation if and only if it is the difference of two positive not decreasing functions in the direction of the positive axes.

If $f(M)$ is of bounded variation and continuous then it is only possible to prove that the function $P(M)$ is continuous along ascending curves. The definition is too restrictive and the procedure is not available for the applications, particularly for the treatment of the area of a surface.

About this topic Tonelli wrote a series of papers on the *Rendiconti dell'Accademia dei Lincei*, culminating in 1926 in a communication to the *Accademy of Sciences de Paris* (Tonelli 1960, 453), where he gave two new definitions of a function of bounded variation and a definition of an absolutely continuous function. These definitions turned out to be very useful also in the studies about the double series of Fourier and in the calculus of variations.

Given the continuous function $f(x,y)$ defined in the square Q whose opposite vertices are $(0,0)$ and $(1,1)$, for every $\bar{x} \in [0,1]$ Tonelli denotes by $V_y(\bar{x})$ the total variation of the function, of the only y , $f(\bar{x},y)$ with y varying in $[0,1]$, with an analogous meaning for the function obtained by exchanging x for y , $V_x(\bar{y})$. The function $f(x,y)$ is said to be of bounded

variation in Q if $V_x(y)$ and $V_y(x)$ are a.e. finite in the interval $[0,1]$ and if the Lebesgue integrals: $\int_0^1 V_{(y)}(x)dx$ e $\int_0^1 V_{(x)}(y)dy$ exist finite.

Tonelli defines the area of a surface as Lebesgue, that is it is the minimum limit of the areas of the polyhedral surfaces tending to the considered surface and proves that the surface of equation $z=f(x,y)$, $(x,y) \in Q$, has finite area if and only if $f(x,y)$ is of bounded variation. If $f(x,y)$ is of bounded variation in Q then the partial derivatives with respect to x , $p(x,y)$, and y , $q(x,y)$, exist finite a.e. in Q , are integrable and for the area of the surface, S , it is:

$$S \geq \iint \sqrt{1 + p^2 + q^2} dx dy,$$

the integral being extended to Q .

In the previous inequality the equality holds if the function $f(x,y)$ is also absolutely continuous, that is the partial functions $f(\bar{x},y)$ e $f(x,\bar{y})$ are absolutely continuous, the former for a.e. \bar{x} , the latter for a. e. \bar{y} in $[0,1]$ and the variations $V_y(\bar{x})$ and $V_x(\bar{y})$ are integrable functions respectively of \bar{x} and \bar{y} , in $[0,1]$.

Observe that if a function is of bounded variation in Arzelà's meaning it is also of bounded variation in the sense now exposed. As a matter of fact for a function satisfying Arzelà's definition the functions $V_{(y)}(x)$ and $V_{(x)}(y)$ are bounded. It is not true the contrary. Moreover if the function is of bounded variation in the sense of Vitali may not be of bounded variation in the sense of Tonelli: indeed the function defined in the unit square by $f(x,y)=g(x)$ where $g(x)$ is not of bounded variation in the interval $[0,1]$, is of bounded variation in the sense of Vitali (actually all the increments $\Delta f=f(x+h,y+k)-f(x+h,y-k)-f(x-h,y+k)+f(x-h,y-k)$ in the partial rectangles are zero) while obviously it is not of bounded variation in the sense of Tonelli. But there are functions of bounded variation in the sense now exposed but not in Vitali's sense (Tonelli 1926 a).

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During the proof Tonelli introduces the concept of total derivative of the function $F(x,y)$ as Vitali: indeed he considers the increment $\Delta F=F(x+h,y+h)-F(x+h,y-h)-F(x-h,y+h)+F(x-h,y-h)$ and defines the derivative as the limit for h tending to zero, if it exists, of the ratio $\Delta F/4h^2$.

Extension of Rolle's Theorem to Additive Set Functions

Only about seven years after he has written the paper *Sui gruppi di punti e sulle funzioni di variabili reali*, Vitali, in that period teacher in a school at Voghera, begins again to produce research papers. He gets in touch with his friend Fubini, who is working about the extension of the mean theorem to additive set functions and gives his contribution to the argument.

In those years there was some interest about this topic. Peano had written a paper, (Peano 1914-15), quoting a surprising treatment of the question by Cauchy in the *Exercices d'Analyse et de Physique mathématique* under the title *Mémoire sur le rapport différentiel de deux grandeurs qui varient simultanément*.²⁹ Peano starts with an observation: if a function of one variable f is derivable in an interval in the more restrictive sense that for every point x_0 of this interval it is

$$f'(x_0) = \lim_{x \rightarrow x_0, y \rightarrow x_0} \frac{f(x) - f(y)}{x - y},$$

then the derivative $f'(x)$ is a continuous function. But, for an additive set function f the derivative is just defined in this way, since, Peano says, $f'(A) = \lim \frac{f(E)}{m(E)}$ when the set E tends to the point A (and not necessarily A belongs to E): so if $f'(A)$ exists finite for every A , then

²⁹ t.2, Paris 1841, pp.188-229.

$f'(A)$ is a continuous function.³⁰ Peano concludes, thank to his very strong definition of derivative:

If a bounded, closed and continuous body (or class of points) has a determined density in every point, then the mean density coincides with the density of the body in a suitable point (Peano 1914-15, 1155).

Also Fubini had dealt with the same argument and in (Fubini 1914-15) had proved the following proposition:

If f is a finitely additive function derivable in all the points of a bounded measurable set J such that if T is a part of J between two parallel lines at a distance of h each from the other then $f(T)$ tends to 0 when h tends to 0, then for this function Rolle's theorem hold, that is if $f(J)=0$ then there exists a point A interior to J such that $f'(A)=0$.³¹

Fubini added in this way in his hypotheses a weak form of continuity for the function that holds for example, he observed, for the integral functions. Here is the elementary proof by Fubini: he considers the domain J enclosed between two right lines $y=a$ and $y=a+h$. For every y he considers the part J_y of J whose points have ordinate not greater than y and he puts $F(y)=f(J_y)$. The function $F(y)$ is continuous by hypothesis, moreover $F(a)=F(a+h)=0$. If F is not identically zero then there exists a point $\eta \in]a, a+h[$ interior to J which is a point of minimum or maximum for F . If $\sigma \leq \frac{h}{4}$ is suitable small, in the interval $] \eta - \sigma, \eta + \sigma [$ there exist two points $y=b$ and $y=b+k$ where F assumes the same value and therefore: $F(b+k)-F(b)=0$. Let T be the part of J between the right lines $y=b$ and $y=b+k$. It is $f(T)=f(J)=0$.

Let us repeat about T the reasoning made about J , exchanging the role of x and y . We obtain a set J_1 interior to J such that $f(J_1)=0$ and each projection of it on both the coordinate axes is less than the half of the analogous projection of J . Repeating for J_1 the previous procedure and continuing indefinitely we determine a sequence of domains $J \supseteq J_1 \supseteq J_2 \supseteq \dots$ such that the projections of every J_n on every one of the two axes is less than the half of the analogous projections of J_{n-1} and $f(J)=f(J_1)=f(J_2)=\dots=0$. There exists one and only one point A belonging to all the domains J_n . Since the diameter of J_n tends to 0 then $f'(A)=\lim \frac{f(J_n)}{m(J_n)}=0$.

If the previous continuity condition was dropped Fubini only was able to prove a weak version of the mean value theorem (Fubini 1915b), namely:

If $f(T)$ is an additive and derivable set function defined in the closed partial domains of a measurable and bounded domain J , if l and L are the g.l.b and l.u.b of the derivative f' in the points A of J , then $l \leq \frac{f(J)}{m(J)} \leq L$. And, if $l < L$ it is $l < \frac{f(J)}{m(J)} < L$.

Then, as a consequence, Fubini proves:³²

If two (finitely) additive and derivable functions $f(T)$ and $g(T)$ have equal (finite) derivatives, they are equal.

³⁰ For the proof see (Peano 1887, 171-172), where additive set functions and their derivatives are introduced in a pioneering way.

³¹ The thesis was a consequence of Peano's result, but Fubini had some doubt to use Peano's definition of derivative, very different from the definitions by Lebesgue and de la Vallée Poussin, as is possible to see by some letters he sent to Vitali (Vitali 1960, 518-520).

³² Fubini's interest for the new mathematical topic is also testified by the new chapter, *Funzioni Additive Generali e Integrali Multipli*, he added to the second edition of his textbook (Fubini 1915a).

Vitali writes two papers on the mean value problem: the first one is a note derived from a letter to Fubini where he shows that the continuity condition of the function used by Fubini can be dropped if the domain is for example a rectangle (Vitali 1915-16). In the second paper (Vitali 1916) he also observes that while in the theorem of Fubini the point A is interior, so that the function can be supposed derivable only in the interior points of the given domain, in his proposition the function has to be supposed derivable in all points of the domain. Here is the theorem established by Vitali:

Let J be a rectangle $ABCD$, let $f(E)$ be an additive function, defined in the class of the measurable subsets E of J , that has a finite derivative (in the sense of Lebesgue) in every point of J and let $f(J)=0$: then there exists a point P belonging to J such that $f'(P)=0$.

The elementary proof is based on the subdivision of the rectangle J in the following way: Vitali divides the edge AB in 2^n equal parts and draws the parallel lines to BC passing through the division points; in the same way he divides BC in 2^n equal parts and draws the parallel lines to AB passing through the division points: in this way for every natural number n he obtains a system S_n of equal rectangles. Since $f(J)=0$, there exist two consecutive rectangles T_1' and T_1'' of S_1 such that $f(T_1') \geq 0$ and $f(T_1'') \leq 0$. Let $T_1 = T_1' \cup T_1''$. Then in T_1 there exist two consecutive rectangles T_2' and T_2'' of S_2 such that $f(T_2') \geq 0$ and $f(T_2'') \leq 0$. Let $T_2 = T_2' \cup T_2''$ and so on. The rectangles T_1, T_2, T_3, \dots have one and only one common point P . It is not difficult to prove now that $f'(P)=0$ and the theorem is proved.

A property like “there exist two consecutive rectangles in which f changes sign” is called in (Peano 14-15, 1153) “semidistributive”: it is a property u in a given class of sets Γ such that if A and B belong to Γ and $A \cup B$ verifies u , then or A verifies u or B verifies u but it is not true the converse. A proof in which a semidistributive property is involved is analogous to the proof of the zero theorem for continuous functions of one variable.

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It is evident that the constructions used by Fubini and Vitali for their proofs are very close to the procedures of the *reseaux* introduced by their Belgian colleague de la Vallée Poussin.

Functions of Bounded Variation

After the absolutely continuous additive set functions, Lebesgue studies in his paper *Sur l'intégration des fonctions discontinues* the functions of bounded variation of more than one variable in an analogous way. As Vitali does, starting from a function of bounded variation of one (or more) variables, he obtains the additive set function corresponding. As Vitali does, if f is a function of two variables, he defines the function $\Psi(E)$.

Lebesgue defines also for such a function the *saltus* in a given point³³ in an analogous way that in the case of one variable. A point is called *singular* if one of the saltus is different from zero.³⁴ He introduces also the singular segments (or varieties in more than two variables). Then he obtains the important result:

A function of bounded variation of k variables gives rise to an additive set function that has a.e. a finite derivative; the derivative is summable and the function coincides with the

³³ There are 4 saltus for a function of two variables.

³⁴ Lebesgue proves that a point is singular if at least one of the two derived numbers is infinite.

sum of an absolutely continuous function, the integral function of its derivative, and a singular part, the saltus function whose derivative is a. e. zero.³⁵

Many years after, in 1922, Vitali, comes back to the early studies about functions of real analysis and being in the dark about preceding results, proves an analogous elegant and more precise proposition, about functions of bounded variation of one variable. Here is the statement:

Every function of bounded variation is the sum of a saltus (discontinuous) function, of an absolutely continuous function and of a linear combination of finite or infinitely many elementary spreads,

where the elementary spreads are continuous functions of bounded variation, not absolutely continuous, as the Cantor function (Vitali 1922 e 1923).

By a letter of March 30th, 1923, Fréchet will acquaint Vitali with the results obtained by de la Vallée Poussin.³⁶

In 1925 Vitali will come back to the argument with the important note *Sulle funzioni continue* (Vitali 1925), where a new remarkable characterization of the continuous functions of bounded variation is exposed, known today under the name of Banach-Vitali theorem since also Banach obtained similar results by means of different methods.

Given a function $f(x)$ defined in an interval (a,b) and with values in (c,d) , let G_∞ be equal to the set of points of (c,d) corresponding to infinitely many points of (a,b) , and, for every natural r , let G_r be equal to the set of points of (c,d) corresponding to precisely r points of (a,b) . Moreover, let F_r be the set of points of (c,d) corresponding to r points at least of (a,b) . Then the F_r are measurable and it is: $G_r = F_r - F_{r+1}$. Vitali proves that the following theorems hold:

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Theorem 1. Let $f(x)$ be a continuous function in (a,b) ; $f(x)$ is of bounded variation if and only if the series $\sum \mu(r)$, where $\mu(r)$ is the measure of F_r , is convergent. In every case the total variation of f is given by the sum of such a series.

³⁵ In (de la Vallée Poussin, 1909) the Author says that he has completely changed the integral part of the treatise with respect to the first edition in order to expose the beautiful works by Lebesgue; in this part (pp.269-272), we find the following theorem quoted in (Lebesgue 1910): "Let $f(x)$ be a continuous function of bounded variation in the interval (a,b) , γ a derivative number of $f(x)$, E the set of the points of (a,b) where γ is infinite; then γ is summable in the set of points where it is finite and $f(b)-f(a)=\int_a^b f(x)dx+V$, where V is the variation of f in E , that is the limit of the sums of the variations in a countable set of disjoint intervals whose union covers E when the length of these intervals tends to zero".

In a footnote de la Vallée Poussin remembers that he had communicated this result to Lebesgue, furnishing the demonstration for the case of a monotone function f . Lebesgue answered immediately, furnishing the general proof, based on the consideration of an intervals chain.

³⁶ We can find the final formulation of the theorem under consideration for the functions of a real variable, jointly with the demonstration, in the second edition of the *Leçons* by Lebesgue, sent to press on December, 1926, in the Chapt.IX:

A bounded variation function $f(x)$ in an interval (a,b) has a finite derivative a. e. and its derivative is summable. Moreover $f(x)$ is the sum of an absolutely continuous function (the integral function of its derivative) and a bounded variation function, the saltus function, having a. e. zero derivative.

Theorem 2. Let m_r be the measure of G_r . Then f is of bounded variation if and only if G_∞ has null measure and the series $\sum r m_r$ converges: in such a case the sum of this series equals the total variation of f .

Arzelà's Contribution to Calculus of Variations

In the 19th century many mathematicians were interested in the Dirichlet problem, concerning the search of a harmonic function u in a tridimensional set E verifying the condition $u = f$ on the boundary of E , S , where f is a given function. Some mathematicians thought to obtain the resolution of Dirichlet problem using the "minimum principle" introduced in Mechanics in the XVIIIth century, that is, given a function f on S , they believed that the solution of the problem was a function assuming the value f on S and minimizing the integral

$$\iiint \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] dx dy dz.$$

This method was called in the sequel Dirichlet's Principle and for many years nobody had doubt about the existence of such a minimum. Many properties of the harmonic functions could be deduced by the study of holomorphic functions. The first who thought to invert this trend and to use the properties of harmonic functions to study holomorphic functions was Riemann. He considered the double integral

$$\iint \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy : \quad (*)$$

since it is positive and varies with continuity he admitted the existence of the minimum and supposing that the boundary conditions were verified, obtained the fundamental theorem of the conformal representation of a simple connected region onto the unitary circle. Weierstrass was the first to observe that there was no prove about the existence of such a minimum and subsequently Prym and Hadamard gave examples of harmonic functions in E , continuous in the closure of E , such that the integral (*) is infinite (Diedonné 1996, 337). From that negative result many researches were born in order to determine under what conditions Dirichlet principle holds, that is when it is possible to establish the existence of the minimum without recourse to the differential equations. It was necessary to investigate a manner to transfer Weierstrass theorem to prove the existence of maxima and minima for continuous functions, to more general situations.

Just in line with the recent studies by Volterra, C. Arzelà, in 1889, in *Funzioni di linee*, and some years after in 1895 in *Sulle funzioni di linee*, considers a class of equi-continuous functions, a distance between functions, a neighborhood of a function and a concept of function that is the uniform limit of a suitable subsequence of an infinite family of functions. He proves, using Ascoli's theorem, that such a limit exists if the functions of the family are uniformly bounded and equi-continuous.

As Cantor Arzelà calls a family G of functions *closed* when every its limit point belongs to G . If F is a real function defined in a closed family of functions G he says that F is continuous if $F(u) = \lim_n F(u_n)$ for every sequence u_n of points of the family G that has u as its limit.

We underline that in Arzelà's definition the function works on *entities, objects* that (as in Volterra) are not completely abstract yet, but may be points or curves (*linee*). Also Volterra defines the concept of continuity for such a type of functions, we now call functionals (after Hadamard), but he does not deal with the possibility of transferring to them Weierstrass theorem. This is instead the intention of Arzelà who wants to use this theorem for applying it to the proof of the Dirichlet Principle:

Theorem – Let F be a real function defined in a family of functions G . Let the functions of the family G be equi-continuous and let G be closed. Then there is in G at least a function $v(x)$ such that the l.u.b. of the values of F in every neighborhood of $v(x)$ coincides with the l.u.b. of F in G . An analogous proposition holds for the g.l.b. of F . If F is continuous then there exist in G two functions U and V such that $F(U) \leq F(u) \leq F(V)$ for every $u \in G$ (Arzelà 1895-96).

It is worth noticing that in a communication presented at the first International Congress of Mathematicians held in Zurich in 1897 J. Hadamard (1865-1963) claimed the importance of functional analysis in the study of Calculus of Variations and in general for the theory of differential equations. In the same year an important step to apply a direct method to the resolution of Dirichlet principle appeared in the paper by Arzelà *Sul principio di Dirichlet*, where for the first time an attempt is made to restore Riemann's demonstration using the theorem of Weierstrass. It is also worth noticing that few years after an analogous generalization of Weierstrass theorem for functionals defined in abstract compact sets would be carried out by Frechet, Hadamard's disciple, in his doctorate thesis (Frechet 1906).

Consider now (Arzelà 1896-97). Given an open set A whose boundary is a continuous curve C , let $f(x,y)$ be a continuous function defined in the points of C .

Arzelà considers the class of functions $u=u(x,y)$, defined in the closure of A such that:

- 1) $u(x,y)$ are continuous with their first and second derivatives,
- 2) the second derivatives and the incremental ratios of the second derivatives are uniformly bounded,
- 3) $u(x,y)=f(x,y)$ in the points $(x,y) \in C$.

Then he proves that the considered class is closed with respect to the uniform convergence and, as he has proved in the preceding paper, there is a function U in the class such that, if

$$J(u) = \iint \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx dy,$$

where the integral is extended to A , then it is $J(U) \leq J(u)$ for every u in the class. Arzelà proves that such a function U is harmonic. Indeed consider a function $U + \alpha V$, where α is whatever real number, positive or negative and V is a function satisfying 1) and 2) that is zero in the points of C . Then by Gauss's formula it is:

$$\begin{aligned} J(U + \alpha V) &= J(U) + \alpha^2 J(V) + 2\alpha \iint \left[\frac{\partial U}{\partial x} \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial V}{\partial y} \right] dx dy \\ &= J(U) + \alpha^2 J(V) - 2\alpha \int V \frac{\partial U}{\partial n} ds - 2\alpha \iint V \Delta U dx dy = J(U) + \alpha^2 J(V) - 2\alpha \iint V \Delta U dx dy \end{aligned}$$

where Δ is the Laplacian. Since it is $J(U + \alpha V) \geq J(U)$ for every α , Arzelà proves that $\iint V \Delta U dx dy = 0$ and, since V is arbitrary and ΔU is continuous, then $\Delta U = 0$.

In this very original way Arzelà proved the existence of the minimum for the integral of the square of the Laplacian only in a subclass of the class of continuous functions. Only three years after, at the Second International Congress of Mathematicians held in Paris, without mentioning Arzelà's work, that he probably did not know, D. Hilbert claimed he was convinced that in order to solve differential equations with partial derivatives in particular as in the classic case of Dirichlet' Principle, some proof of existence could be produced on the basis of a general minimizing criterion. In particular he considered the problem of minimal surfaces: it consists in the search, among all the surfaces that have a given boundary, of the surface with the smallest area, that is such that, given a domain D , the integral, extended to D ,

$$\iint \sqrt{1 + \left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2} dx dy$$

is minimum. In the past the problem, dealt also by Lagrange, was reduced to a differential equation, the Euler- Lagrange equation, but, among other difficulties linked essentially with the fact that the theory of surfaces was not completely advanced, it was difficult to verify whether the solution of the equation was actually a minimum or not. In his relation Hilbert claimed that this problem and Laplace equation were particular cases of the following general problem: determine a function $z=f(x,y)$ for which the integral:

$$\iint F(x, y, z, p, q) dx dy, \text{ where } p=\frac{\partial f}{\partial x}, q=\frac{\partial f}{\partial y}, \text{ is minimum under the condition } \frac{\partial^2 F}{\partial p^2} \frac{\partial^2 F}{\partial q^2} - \left(\frac{\partial^2 F}{\partial p \partial q}\right)^2 > 0.$$

Hilbert wondered whether the solutions of such problems were always analytical, a question that has been answered negatively in the sequel.

Tonelli's Contribution to Calculus of Variations

The vast amount of work and papers Tonelli produced on Calculus of Variations from 1911 is strictly linked to the works of Arzelà (his professor at the University of Bologna) and Hilbert. We will start with one of his most important papers of the early period, the Memoir published in 1915 *Sur une méthode directe du calcul des variations*, (Tonelli 1915) where the results exposed in two notes appeared on the C.R. of the Acc. of the Sciences of Paris the year before are assembled. In it Tonelli remembers that Arzelà had tried to apply the direct method in order to solve the Dirichlet's problem: his efforts were in vain but his direct method was valid since the double integral of the square of the gradient even if not continuous is lower semi-continuous. Tonelli underlines that Hilbert was the first to define correctly the problem: he had proved in 1900 that the technic deduced from the theory of continuous functions allowed really to solve Dirichlet problem (Hilbert 1900). But Hilbert, in the opinion of Tonelli, did not catch the very and deep essence of the method. He proved actually that a minimizing sequence of surfaces, chosen among the admitted ones, such that the respective integrals tended toward the g.l.b. that had to be proved to be the limit, had at least a limit surface. By the use of an already known solution in a particular case of the same problem, he proved that the limit surface was a solution of the problem. Then he never used the fact that, given the minimizing sequence, the functional tends to the value assumed in the limit surface, that is so the required minimum value. In this way he never refers explicitly to continuity, that is the property that allows the passage to the limit. And the field of applications of his method remains in this way bounded. This limitation was underlined also by J. Hadamard who furnished another direct method in order to prove the existence of the minimum. Given the surface of equation $z=f(x,y)$ and a point M on it, he affirms that if one starts from M and follows the maximum gradient line in the sense of decreasing z , he arrives at a minimum point. From this idea Hadamard deduces a procedure of successive approximations which allows a method for determining effectively the minimum (Hadamard 1906). But also Hadamard finds some difficulties with this method.

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As in the preceding papers, Tonelli is sure that in general the method valid for the ordinary continuous functions can be actually extended to functional calculus under suitable hypotheses and can be completed in order to give the analytical properties of the solution. He underlines the contribution that it is possible to obtain from Baire's idea to introduce the notions of lower and upper semi-continuity, in order to obtain a sufficient condition for the existence of the minimum and the maximum.

But there is a difficulty: every infinite sequence of points of a bounded interval (a,b) has at least one accumulation point; this fact is not true if we consider a family of functions. In this case it is necessary to add some supplementary conditions in order to apply some particular result as Ascoli's theorem. Once the existence of the minimizing element will be

proved, those supplementary conditions will imply also the analytical properties of the solution.

Let us show how Tonelli applies the direct method to the functional:

$$J(y) = \int_a^b f(x, y(x), y'(x)) dx \quad (*)$$

where A is a set of points (x, y, y') such that: $a \leq x \leq b$, $-\infty < y < +\infty$, $-\infty < y' < +\infty$; and $f(x, y, y')$ is a continuous function with $f_{y'}, f_{y'y'}, f_{y'x}$ in A such that:

I) $f_{y'y'}(x, y, y') \geq 0$ in A ;

II) there exist constants $\alpha > 0$, $h > 0$ and k such that $f(x, y, y') \geq h |y|^{1+\alpha} + k$.

The problem linked to (*) is: determine in the class F of absolutely continuous functions $y(x)$ such that $f(x, y(x), y'(x))$ is integrable a minimum of $J(y)$.

Tonelli places himself in the set of the conditions of the regular problems (so called after Hilbert).

In general the integral in (*) is called *positively (negatively) regular* if the invariant $f_{y'y'}(x, y, y')$ is > 0 (< 0) for every $(x, y, y') \in A$. It is *almost positively (negatively) regular* if, as in the condition I) and II), $f_{y'y'} \geq 0$ (≤ 0) and $f > 0$ (< 0).

Tonelli proved for the first time in 1913 in the paper *Sul caso regolare nel calcolo delle variazioni*, that positive (negative) regularity implies the lower (upper) semi-continuity of the functional. In a brief paper in 1914, Tonelli proves that if the functional is continuous then the Euler equation $f_y - \frac{df_{y'}}{dx} = 0$, that is the condition a minimum of the functional has to satisfy in general, reduces to $f_{y'y'} = 0$ and therefore $f(x, y, y') = P(x, y) + Q(x, y)y'$ a very restrictive condition implying that all those problems that really concern the calculus of variations are excluded (Tonelli 1914). In this paper Tonelli expresses also the condition that the functional is considered in the class of absolutely continuous functions (firstly he has considered the functional defined in the class of continuous rectifiable curves); in *Fondamenti*, he insists on the semi-continuity of the almost regular integral (*) in the class F of the absolutely continuous functions $y(x)$ in (a, b) such that $f(x, y(x), y'(x))$ is integrable, a larger class than the class of the continuous functions with their first derivatives, used in the classical theory. Lavrentieff will prove that it is not possible to enlarge the class of absolutely continuous functions since if one considers in particular the class of the functions of bounded variation as admissible functions then the g.l.b. of the integral in this class coincides with the g.l.b. in the class of all measurable functions: and it is achieved in a pair of measurable functions $g(x)$ and $h(x)$, but in general $h(x)$ is not the derivative of $g(x)$ (Lavrentieff 1926-27).

Let us come back to the previous paper (Tonelli 1915): in order to prove that the functional (*) admits a minimum in F , as usual Tonelli proves firstly that J is a lower semi-continuous functional in F . This is proved using the fact that if $f_{y'y'} \geq 0$ in A then the Weierstrass function

$$E(x, y; y', y'_0) = f(x, y, y') - f(x, y, y'_0) - (y' - y'_0) f_{y'}(x, y, y'_0)$$

is ≥ 0 for every $(x, y, y') \in A$ and $(x, y, y'_0) \in A$.

Then given two points $P=(a, p)$ and $Q=(b, q)$ he proves the following:

Theorem 1– If $f(x, y, y')$ satisfies conditions I) and II), there exists in the class F a function $y_0(x)$ such that the curve of equation $y=y_0(x)$ has extreme points P and Q and minimizes integral (*).

Sketch of the proof: let $L = \inf J(y)$ in the class of the functions of F whose extreme points are P and Q . By II) L is finite. It is possible to obtain, without the use of the Axiom of Choice as Tonelli has proved explicitly in (Tonelli 1913), a sequence of functions of F , $y_n(x)$, such that $y_n(a) = P$, $y_n(b) = Q$ and $J(y_n) < L + 1/n$ for every natural number n . The functions y_n are uniformly bounded and Tonelli proves that they are also equi-continuous so it is possible to apply Ascoli's theorem and therefore there exists a subsequence, we can still call $y_n(x)$, which is uniformly convergent. Let $y_0(x)$ be the limit function of $y_n(x)$: it is absolutely continuous, $y_0(a) = P$, $y_0(b) = Q$. Since J is lower semi-continuous it is $\liminf J(y_n) = J(y_0) = L$ and therefore y_0 is the required minimum. Moreover:

Under suitable hypotheses on f , y_0 is derivable in $[a, b]$ with continuous derivative, satisfies Euler equation and is the unique solution.

Tonelli wrote a series of important works about the calculus of variations that would continue to produce during all his life and that would culminate with the fundamental treatise (Tonelli 1921-23).

In 1929 obtains an interesting extension of his direct method to the double integral:

$$I_D = \iint f(x, y, z, p, q) dx dy \quad (**)$$

where z is a function of two variables, $z = z(x, y)$, p and q are its partial derivatives and the integral is extended to a plane domain D .

Tonelli remembers that E. Goursat (1858-1936) in 1915 established that (**) is lower semi-continuous in the class of the functions continuous with their first order derivatives if there exist two constants $\alpha > 0$ and $\beta > 0$ such that:

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$$u^2 f_{pp} + 2uv f_{pq} + v^2 f_{qq} \geq \alpha u^2 + \beta v^2 \quad \text{for every admissible } (x, y, z, p, q),$$

a condition that implies Hilbert condition of regularity; in 1927 R. Rado (1906-1989) proved that the integral (**), where $f(x, y, z, p, q) = \sqrt{1 + p^2 + q^2}$, is lower semi-continuous in the class of the functions satisfying a Lipschitz condition, that is such that for every function z there exists a constant $L(z) > 0$ such that

$$|z(x_1, y_1) - z(x_2, y_2)| \leq L(z) \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad \text{for every } (x_1, y_1), (x_2, y_2) \in D;$$

and in 1926 Haar gave a more general result considering arbitrary functions $f(p, q)$ such that the integral (**) is positively regular, that is $f_{pp} f_{qq} - f_{pq}^2 > 0$, $f_{pp} > 0$.

In 1929 in the paper *Sur la semi-continuité des intégrals double du calcul des variations*, Tonelli proves the semi-continuity of the integral (**) by means of a very general method (Tonelli 1929): he supposes in fact $f(x, y, z, p, q)$ continuous with its partial derivatives $f_p, f_q, f_{pp}, f_{qq}, f_{pq}, f_{px}, f_{qy}$ in every point (x, y) of the open and bounded set D and for every finite z, p, q ; as usual the integral (**) is called *almost positively regular* if $f_{pp} f_{qq} - f_{pq}^2 \geq 0$, $f_{pp} \geq 0$, $f_{qq} \geq 0$ for every $(x, y) \in D$ and for every z, p, q .

Tonelli considers also the most general functions $z(x, y)$ that it was possible conceive in this frame, that is the absolutely continuous functions of two variables as he himself has defined in *Sur la quadrature des surfaces*, obviously only the ones such that the function $f(x, y, z, p, q)$ is integrable. Let F be the class of such functions. Then the following holds:

Theorem 2– Suppose that:

- a) there exists a real number N such that $f(x,y,z,p,q) > N$ for every (x,y) in D and finite z, p and q ;
- b) the integral I_D is almost positively regular.

Then I_D is lower semi-continuous in F (Tonelli 1929, &13).

In the paper *L'estremo assoluto degli integrali doppi* (Tonelli 1933) the Author exposes the continuation of the preceding research. In it indeed Tonelli searches to determine the minimizing functions of the integral (***) and their degree of regularity.

In order to carry out this program Tonelli establishes an interesting condition a family of functions has to satisfy in an open and bounded set in order to be equi-continuous:

Theorem 3 - Suppose that the functions $\{f(x,y)\}$ of a given family are all absolutely continuous in a bounded open set A and there exist two numbers $p > 2$ and $L > 0$ such that for every function of the family the double integral extended to A : $\iint \left(\left| \frac{\partial f}{\partial x} \right|^p + \left| \frac{\partial f}{\partial y} \right|^p \right) dx dy$ is less than L . Then the functions of the family are equi-continuous in A .

To give a result of equi-continuity up to the boundary Tonelli introduces a property the boundary of the set A has to satisfy: it is a rather difficult condition to expose and we limit ourselves to one of its consequences, the cone property in two dimensions that can be defined in the following way:

- γ) there exists an angular sector T such that for every point P on the boundary of A , P is the vertex of an angular sector similar to T whose interior points are all contained in A . In particular γ) holds if the boundary of A is a regular curve.

Tonelli begins by proving the following theorem of prolongation by continuity:

Theorem 4 - Let the function $f(x,y)$ be absolutely continuous in a bounded open set A ; suppose that there exists a number $p > 2$ such that the double integral extended to A :

$$\iint \left(\left| \frac{\partial f}{\partial x} \right|^p + \left| \frac{\partial f}{\partial y} \right|^p \right) dx dy$$

is finite and suppose that γ) holds: then it is possible to prolong the given function to a continuous function in the close set \bar{A} .

Tonelli proves that if γ) does not hold, even if the other conditions are satisfied, the prolongation up the boundary may be not possible. He exhibits a:

Counterexample - Define the open set A and the function $f(x,y)$ in the following way: let the strictly increasing sequence $0 \leq a_0 < a_1 < a_2 < \dots < a_n, \dots < \pi$ be given in such a way that $t_0 = a_1 - a_0 = 2^{-2p}$, $t_n = a_{n+1} - a_n = 2^{-(n+2)p}$ for every natural number n . Consider a polar coordinate system whose pole is in the origin and polar axis is the positive semi-axis of (x,y) plane. Let A be the open set that is the union of the following sets:

$$A_n = \{(\rho, \vartheta) : 2^{-(n+1)} \leq \rho \leq 2^{-n}, \quad a_n \leq \vartheta \leq a_{n+1}\}.$$

Let $f(x,y) = g(\rho)$ where

$$g(\rho) = 2^{n+2}(\rho - 2^{-(n+1)}) \quad \text{if } 2^{-(n+1)} \leq \rho \leq 2^{-(n+1)} + 2^{-(n+2)}$$

$$g(\rho) = 2^{n+2}(2^{-n} - \rho) \quad \text{if } 2^{-(n+1)} + 2^{-(n+2)} \leq \rho \leq 2^{-n}.$$



It is easy to prove that:

$$\left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right]^{\frac{p}{2}} = \left| \frac{\partial f}{\partial \rho} \right|^p \quad \text{and therefore:}$$

$$\iint \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right]^{\frac{p}{2}} dx dy = \sum_{n=0}^{\infty} t_n \int_{2^{-(n+1)}}^{2^{-n}} 2^{p(n+2)} \rho d\rho = 1/2.$$

Since if $a > 0$ and $b > 0$ then $a^p + b^p \leq 2(a^2 + b^2)^{p/2}$, we have that $\iint \left(\left| \frac{\partial f}{\partial x} \right|^p + \left| \frac{\partial f}{\partial y} \right|^p \right) dx dy \leq 1$.

Therefore, since $f(x,y)$ is absolutely continuous in A , the conditions of Theorem 4 except γ) are satisfied. The function $f(x,y)$ can be prolonged by continuity to all the points of the boundary different from the origin, but it is not possible to prolong by continuity $f(x,y)$ in the origin. Indeed on the arcs of equation $\rho = 2^{-n}$ it is $f(x,y) = 0$, while on the arcs of equation $\rho = 2^{-(n+1)}$ it is $f(x,y) = 1$. On the other hand condition γ) cannot be satisfied in the origin.

Tonelli proves many theorems about the existence of the minimum of the integral (**). We consider for example the following one where a complete class in (the interior of) the domain D with respect to the integral I_D is considered: it is a class C (C_i) of functions $z(x,y)$ absolutely continuous in (the interior of) D such that for every one of them the integral $I_D(f)$ is finite and every function $g(x,y)$ that is a limit function for the class (with respect to uniform convergence) belongs to the class. From Theorem 3 and Theorem 4 easily follows:

Theorem 5 – Let $I_D(z)$ be an almost positively regular integral extended to the domain D and let $p > 2$, $M > 0$ and N be such that:

$$f(x,y,z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) > M \left\{ \left| \frac{\partial z}{\partial x} \right|^p + \left| \frac{\partial z}{\partial y} \right|^p \right\} + N.$$

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Then if the functions $z(x,y)$ of a complete class C_i are uniformly bounded in the interior of D , there exist in C_i the absolute minimum of $I_D(z)$. If the domain is enough regular, for example if γ) holds, and if the functions constitute a complete class and are uniformly bounded on the boundary of D , then the absolute minimum is continuous also in the closure of D . In few years Tonelli's contribution begins fundamental for the theory at an international level.

Enthusiastic comments to his work were given by Hadamard, Carathéodory, Volterra e Peres (see the Introduction to (Opere scelte) by Silvio Cinquini).

Conclusions

As we have seen, between 1906 and 1915, great strides were made towards the extension of many fundamental concepts of the just arisen measure and function theory. This is a general feature of this period with respect to mathematical analysis and geometry; great changes are introduced particularly by M. Fréchet and D. Hilbert although motivated by different interests.

While, as we have seen, the mathematicians have looked so far after extending the results about the functions of one variable to functions of a finite number of variables, Fréchet (1878-1973) in his PhD thesis suddenly introduces abstract spaces of points (Fréchet 1906). The measure theory is no more linked to Euclidean spaces and Lebesgue theory, but abstract metric spaces are introduced, and Fréchet proves the theorems in a so general way which allows their application in many different particular cases. As we have seen in the preceding Sections already some time before in the problems of variation calculus and

mathematical physics some functionals, that is quantities not depending on a numerical variable, but on a function or on a curve, were considered, the functionals used by Volterra, Arzelà and Hadamard, and also studied in a pioneering manner by Pincherle (Pincherle 1901), being of such a type; but now Hilbert observes that a function, let it be continuous for example, is determined by infinitely many variables, its Fourier coefficients, for example, in such a way that a functional can be viewed as a function of infinitely many real variables. Fréchet in 1910 explains that only after the definition of abstract metric space he gave in his doctoral thesis, Hilbert, motivated by the study of integral equations, was able to give a definition of a space whose points are the real number sequences such that the series of their squares is convergent.

Vitali and Lebesgue, generalizing the functions of bounded variation to the case of more variables, were very near to the general concept of measure that Johan Radom (1887-1956) introduces in 1913: indeed this author, always referring to the family of Lebesgue measurable sets, demonstrates that it is possible to define a Lebesgue type integral with respect to whatever completely additive measure function defined in the Lebesgue measurable sets of the space of one or more dimensions³⁷. Without doubt he was also inspired by the beautiful representation theorem published by Riesz in 1909 (Bourbaki 1960), that is the following theorem:

For every continuous linear functional A on the space C of continuous functions in an interval (a,b) , there exists a bounded variation function α such that A coincides with the Stieltjes integral determined by α , that is $A(f) = \int_a^b f(x)d\alpha(x)$ for every $f \in C$.

The final generalization, where there is no reference to the Lebesgue measurable sets, is dated 1915 and is due to Fréchet who considers in (Fréchet 1915) the integral of a functional extended to an abstract space X on which a family of sets, closed with respect to enumerable unions and to relative complements (*famille additive d'ensembles*, σ -ring today), is given. After, in 1930, Nicodym will complete the new theory: in his paper (Nicodym 1930) he exposes the theory of measure and integration as today is exposed in many books for first graduate students in Mathematics: differently from Fréchet he prefers to use the notion of *corps d'ensembles* (σ -algebra today, that is a σ -ring such that X belongs to it) rather than *famille additive d'ensembles*, and positive measure. He explains that the two choices he does are linked in his theory of integration. Indeed if \mathfrak{I} is a σ -ring and $\mu(E) \geq 0$ is a *parfaitement additive* (σ -additive today) bounded function on \mathfrak{I} and if M is its upper bound then there exists a not decreasing sequence $E_1, E_2, \dots, E_n, \dots$ of elements of \mathfrak{I} such that $\lim \mu(E_n) = M = \mu(\cup E_n)$. Let $E_0 = \cup E_n$. Now if E is such that $E \cap E_0 = \emptyset$ then $\mu(E) = 0$. Thus the sets outside E_0 do not play any role in the theory of integration and therefore it is just the same to start with a σ -algebra.

With these choices he develops in a very simple manner the theory of integration (starting from the definition of integral for simple functions, that is functions that assume at most infinitely many enumerable values, each in a set of \mathfrak{I}), and finally he generalizes completely the theorem first enunciated and proved by Vitali in 1905 for one variable and generalized by Lebesgue for the case of more real variables in (Lebesgue 1910) in the following form:

Given a σ -algebra \mathfrak{I} , a positive measure μ on it and a σ -additive function, $F(E)$, absolutely continuous with respect to μ , there exists a summable function f such that $F(E) = \int_E f d\mu$ for every $E \in \mathfrak{I}$.

³⁷ J. Radon, *Theorie und Anwendungen der absolut additiven Mengenfunktionen*, Sitzungster. der math. naturwiss. Klasse der Akad. der Wiss. (Wien) t.CXXII, Abt. II a (1913), pp.1295-1438.

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